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Spin Effect and Stokes Phenomenon for Fermion Production in Electric Fields

Rong-Gen Cai*
Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

Sang Pyo Kim†
Department of Physics, Kunsan National University, Kunsan 54150, Korea
Center for Relativistic Laser Science, Institute for Basic Science, Gwangju 61005, Korea
Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

Won Kim‡
Department of Physics, Kunsan National University, Kunsan 54150, Korea

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One of the intriguing features of (Sauter-)Schwinger pair production is the Stokes phenomenon of no-pair production in some bipolar electric fields. The spin effect is studied on fermion pair production in the same electric fields of New Physics (Sae Mulli) 68, 1225 (2018), in which pair production of bosons is explained in terms of polons, simple poles with complex frequency. The Stokes phenomenon is showed to occur for specific parameters when the electric field is bipolar and changes the direction of antisymmetry in time. We compare the pair production of bosons and fermions and discuss its physical implications.

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I. INTRODUCTION

One of the intriguing aspects of (Sauter-)Schwinger pair production of electron-positron pairs [1, 2] is the Stokes phenomenon, in which pairs vanish for some profiles of electric fields in the future infinity. Though it is expected that pairs are produced by time-dependent electric fields, it is shown in Ref. [3] that bosons cannot be produced in a bipolar electric field $E(t) = E_0 \sinh(t/\tau)/\cosh^2(t/\tau)$ when $qE_0\tau^2 = (2n+1)/2$ for any integer n. The electric field and its gauge potential with natural number n lead to a reflectionless scattering over the potential barrier for the Klein-Gordon equation, which results in no pair production and is related to the soliton number of the Korteweg-de Vries (KdV) equation [4]. The interferences and Stokes phenomena of pair production were studied in time-dependent electric fields [5–7].

The purpose of this paper is to investigate the spin effect on fermion pair production in the same field configurations of Ref. [3], in which the pair production of bosons exhibits the Stokes phenomenon. The spin-diagonal component of the Dirac equation in a time-dependent electric field $E_\parallel(t)$ with the vector potential $A_\parallel(t)$ (in units of $\hbar = c = 1$) is

$$\ddot{\phi}_{\vec{k}\sigma}(t) + \omega^2_{\vec{k},\sigma}(t)\phi_{\vec{k}\sigma}(t) = 0, \tag{1}$$

where $\sigma = \pm 1/2$ and

$$\omega_{\vec{k},\sigma}(t) = \sqrt{\mu^2 + \left(k_\parallel - qA_\parallel\right)^2 + 2i\sigma qE_\parallel}, \quad \mu^2 = m^2 + \mathbf{k}^2_\perp. \tag{2}$$

Note that the frequency (2) becomes complex due to the imaginary spin components in contrast to the real spin components in a magnetic field. The spinless boson is the limiting case of $\sigma = 0$.

*E-mail: cairg@itp.ac.cn
†E-mail: sangkim@kunsan.ac.kr
‡E-mail: dreamwalker@kunsan.ac.kr

The bosonic pair production or particle production is given by the in-in formalism [8–10]

$$\mathcal{N}_{\mathbf{k}}^{\rm sc} = \left| \sum_{J} \exp\Big[-\frac{i}{2} \oint_{C_J^{(1)}} \big(\omega_{\mathbf{k}}(z) + \omega_{-\mathbf{k}}(z)\big) dz \Big] \right|, \quad (3)$$

where $C_J^{(1)}$ denote all independent contours of winding number one enclosing simple poles. The formulation (3) is another complex analysis of the phase-integral formulation, where the average of the positive and negative momentum is taken [11,12]. We now extend the bosonic formulation (3) to the fermionic formulation

$$\mathcal{N}_{\mathbf{k}}^{\rm sp} = \frac{1}{2} \left| \sum_{J,\sigma} \exp\Big[-\frac{i}{2} \oint_{C_J^{(1)}} \big(\omega_{\mathbf{k},\sigma}(z) + \omega_{-\mathbf{k},\sigma}(z)\big) dz \Big] \right|. \quad (4)$$

Here, the factor $1/2$ is the average over spin components.

The organization of this paper is as follows. In Sec. II, we apply the phase-integral formulation to find the spin effect on the Schwinger pair production in a Sauter electric field and compare it with the exact result for fermions. In Sec. III, we find the fermion pair production formula in an unipolar electric field of hyperbolic type whose direction does not change in time. In Sec. IV, we find the fermion pair production formula in a bipolar electric field of hyperbolic type whose direction changes in time. The solitonic nature of the Schwinger effect is explained via the Stokes phenomena. Finally, we compare the bosonic pair production with the fermionic pair production and discuss the physical implications.

II. Spin effect in Sauter electric field

The electric field configurations that lead to the exact solutions of the Dirac or Klein-Gordon equation are limited and their solutions are classified, for instance, in Ref. [13]. The most well-know configuration in strong field quantum electrodynamics (QED) is the Sauter electric field [1]

$$E_{\parallel}(t) = \frac{E_0}{\cosh^2(t/\tau)}, \quad A_{\parallel}(t) = -E_0 \tau \tanh(t/\tau). \quad (5)$$

The Schwinger pair production with the same spin multiplicity is for spinless bosons [14]

$$\mathcal{N}_{\mathbf{k}}^{\rm sc} = \frac{\cosh\big(2\pi\lambda^{\rm sc}\big) + \cosh\big(\pi\tau(\omega_{\mathbf{k}}(t=\infty) - \omega_{\mathbf{k}}(t=-\infty))\big)}{2\sinh\big(\pi\tau\omega_{\mathbf{k}}(t=\infty)\big)\sinh\big(\pi\tau\omega_{\mathbf{k}}(t=-\infty)\big)}, \quad (6)$$

and for spin-1/2 fermions [15]

$$\mathcal{N}_{\mathbf{k}}^{\rm sp} = \frac{\cosh\big(2\pi\lambda^{\rm sp}\big) - \cosh\big(\pi\tau(\omega_{\mathbf{k}}(t=\infty) - \omega_{\mathbf{k}}(t=-\infty))\big)}{2\sinh\big(\pi\tau\omega_{\mathbf{k}}(t=\infty)\big)\sinh\big(\pi\tau\omega_{\mathbf{k}}(t=-\infty)\big)}, \quad (7)$$

where

$$\lambda^{\rm sc} = qE_0\tau^2 \sqrt{1 - \frac{1}{\big(2qE_0\tau^2\big)^2}}, \quad \lambda^{\rm sp} = qE_0\tau^2, \quad (8)$$

and

$$\omega_{\mathbf{k}}(t=\pm\infty) = \sqrt{\mu^2 + \big(k_{\parallel} \pm qE_0\tau\big)^2}. \quad (9)$$

The one-loop effective actions consistent with the mean numbers (6) and (7) were obtained by the gamma-function regularization in Ref. [16].

We now apply the phase-integral formulation (4) to the Sauter electric field (5). Under a conformal mapping $z = e^{t/\tau}$, the symmetrized relativistic WKB action (4) takes the form

$$\mathcal{S}_{\mathbf{k},\sigma=\pm\frac{1}{2}} = \frac{i}{2}\tau \oint_{C_J^{(1)}} \frac{dz}{z(z^2+1)} \Big[\sqrt{\mu^2(z^2+1)^2 + \big[k_{\parallel}(z^2+1) + qE_0\tau(z^2-1)\big]^2 \pm 4iqE_0 z^2} + \sqrt{\mu^2(z^2+1)^2 + \big[k_{\parallel}(z^2+1) - qE_0\tau(z^2-1)\big]^2 \pm 4iqE_0 z^2} \Big]. \quad (10)$$

There are three simple poles: $z=\infty$ and $z=\pm i$. Then, any loop neither enclosing the finite simple poles $z=\pm i$ nor enclosing them receives a residue contribution from the simple pole at $z=\infty$

$$\mathcal{S}_{\mathbf{k},\sigma=\pm\frac{1}{2},(\infty)} = \pi\tau\big[\omega_{\mathbf{k},\sigma=\pm\frac{1}{2}}(t=\infty) + \omega_{\mathbf{k},\sigma=\pm\frac{1}{2}}(t=-\infty)\big]. \quad (11)$$

On the other hand, the finite simple poles at $z=\pm i$ give the same residue

$$\mathcal{S}_{\mathbf{k},\sigma=\pm\frac{1}{2},(\pm i)} = -\pi qE_0\tau^2 \sqrt{1 \mp \frac{i}{qE_0\tau^2}}, \quad (12)$$

which for $qE_0\tau^2 \gg 1$ takes

$$\mathcal{S}_{\mathbf{k},\sigma=\pm\frac{1}{2},(\pm i)} \approx -\pi qE_0\tau^2 \pm i\frac{\pi}{2}. \quad (13)$$

Therefore, the mean number of produced fermions is

$$\begin{aligned} \mathcal{N}_{\mathbf{k}} &\approx \frac{1}{2} e^{-\pi\tau\left(\omega_{\mathbf{k}}(t=\infty)+\omega_{\mathbf{k}}(t=-\infty)\right)} \\ &\quad \times \left| 2\big(1 + e^{\pi qE_0\tau^2 - i\frac{\pi}{2}} + e^{\pi qE_0\tau^2 + i\frac{\pi}{2}} + e^{2\pi qE_0\tau^2}\big) \right| \\ &= e^{-\pi\tau\left(\omega_{\mathbf{k}}(t=\infty)+\omega_{\mathbf{k}}(t=-\infty)\right)} \big(1 + e^{2\pi\lambda^{\rm sp}}\big). \end{aligned} \quad (14)$$

Note that the phase-integral formulation (14) in the Sauter electric field gives the same pair production for bosons and fermions in the phase-integral formulation, which is the leading terms of Eqs. (6) and (7).

Now, we estimate the spin effect on production of electron-positron pairs for zero-momentum ($\mathbf{k} = 0$) in the Sauter electric field (5) and compare Eq. (7) with Eq. (6) and the fermionic result (14) with the bosonic result in Ref. [3]. The electric field (5) with a time scale τ for the duration of the electric field is characterized by two dimensionless parameters:

$$\epsilon = \frac{m}{qE_0\tau}, \quad \delta = \frac{qE_0}{\pi m^2}. \tag{15}$$

So, for the Schwinger or critical field for the peak intensity $E_0 = E_{\rm S} = m^2/e$, the parameters are determined by the ratio of the electron Compton time $t_{\rm C} = [\hbar]/m[c^2] = 2.06\times10^{-22}s$ to τ for the pulse range from zepto (10^{-21}) to femto (10^{-15}) seconds,

$$2.06\times10^{-7} \leq \epsilon = \frac{t_{\rm C}}{\tau} \leq 2.06\times10^{-1}, \quad \delta = \frac{1}{\pi}. \tag{16}$$

Then, we express quantities in Eqs. (6) and (7) in terms of ϵ as

$$\lambda^{\rm sc} = \frac{1}{\epsilon^2}\sqrt{1-\frac{\epsilon^4}{4}}, \quad \lambda^{\rm sc} = \frac{1}{\epsilon^2}, \tag{17}$$

and $\omega_{\mathbf{k}=0}(t=\infty) = \omega_{\mathbf{k}=0}(t=-\infty)$ with

$$\tau\omega_{\mathbf{k}=0}(\infty) = \frac{1}{\epsilon}\sqrt{1+\frac{1}{\epsilon^2}}. \tag{18}$$

We compare Eq. (7) with Eq. (6) for the range (16)

$$\mathcal{N}^{\rm sc}_{\mathbf{k}=0} = \frac{\cosh\left(2\pi\lambda^{\rm sc}\right)+1}{2\sinh^2\left(\pi\tau\omega_{\mathbf{k}=0}(\infty)\right)}, \quad \mathcal{N}^{\rm sp}_{\mathbf{k}=0} = \frac{\cosh\left(2\pi\lambda^{\rm sp}\right)-1}{2\sinh^2\left(\pi\tau\omega_{\mathbf{k}=0}(\infty)\right)} \tag{19}$$

Due to $\epsilon \ll 1$, the leading terms are, respectively,

$$\mathcal{N}^{\rm sc}_{\mathbf{k}=0} \approx 1 + 4e^{-\frac{2\pi}{\epsilon}}, \quad \mathcal{N}^{\rm sp}_{\mathbf{k}=0} \approx 1. \tag{20}$$

III. Unipolar electric field

The unipolar and time-symmetric electric field [3]

$$E_\parallel(t) = \frac{E_0\cosh(t/\tau)}{\sinh^2(t/\tau)}, \quad A_\parallel(t) = \frac{E_0\tau}{\sinh(t/\tau)}, \tag{21}$$

gives the symmetrized relativistic WKB action from the spin-diagonal frequency (2)

$$\begin{aligned}\mathcal{S}_{C_J^{(1)}} &= \frac{i}{2}\tau\oint_{C_J^{(1)}}\frac{dz}{z(z^2-1)} \\ &\Big[\sqrt{\mu^2(z^2-1)^2+\left[k_\parallel(z^2-1)-2qE_0\tau z\right]^2 \pm 2iqE_0z(z^2+1)} \\ &+\sqrt{\mu^2(z^2-1)^2+\left[k_\parallel(z^2-1)+2qE_0\tau z\right]^2 \pm 2iqE_0z(z^2+1)}\Big],\end{aligned} \tag{22}$$

where we used a conformal mapping $z = e^{t/\tau}$ in the second line. We assume that proper branch cuts can make the square root an analytical function of z. There are three simple poles in the complex z-plane: $z=\infty$ and $z=\pm1$. The residue contribution from $z=\infty$ is

$$\mathcal{S}_{\mathbf{k},\sigma=\pm\frac{1}{2},(\infty)} = 2\pi\tau\omega_{\mathbf{k}}(z=\infty), \tag{23}$$

while the contributions for each spin component from the poles at $z=\pm1$ are

$$\begin{aligned}\mathcal{S}_{\mathbf{k},\sigma=\frac{1}{2},(\pm1)} &= 2\pi qE_0\tau^2\sqrt{1\pm\frac{i}{qE_0\tau^2}} \approx 2\pi qE_0\tau^2 \pm i\pi, \\ \mathcal{S}_{\mathbf{k},\sigma=-\frac{1}{2},(\pm1)} &= 2\pi qE_0\tau^2\sqrt{1\mp\frac{i}{qE_0\tau^2}} \approx 2\pi qE_0\tau^2 \mp i\pi.\end{aligned} \tag{24}$$

Adding the contributions of spins and independent loops of winding number one around $z=\pm1$ and noting that the contribution from the simple pole at $z=\infty$ is a common factor to the contributions enclosing any finite simple pole, we obtain

$$\begin{aligned}N^{\rm sp}_{\mathbf{k}} &\approx \frac{1}{2}e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)}\left|2\left(1-2e^{-2\pi qE_0\tau^2}+e^{-4\pi qE_0\tau^2}\right)\right| \\ &= 4e^{-2\pi\tau(\omega_{\mathbf{k}}(t=\infty)+qE_0\tau)}\sinh^2\left(\pi qE_0\tau^2\right).\end{aligned} \tag{25}$$

The pair production of fermions (25) has $\sinh^2\left(\pi qE_0\tau^2\right)$, which differs from $\cosh^2\left(\pi qE_0\tau^2\right)$ for boson pair production in Ref. [3]. As expected, the unipolar electric field does not exhibit destructive interference for pair production because the time-symmetric potential energy is positive and provides virtual pairs with enough potential energy to separate them into real pairs.

IV. Bipolar electric field

The time-asymmetric bipolar electric field [3]

$$E_\parallel(t) = \frac{E_0\sinh(t/\tau)}{\cosh^2(t/\tau)}, \quad A_\parallel(t) = \frac{E_0\tau}{\cosh(t/\tau)}, \tag{26}$$

can be obtained by analytically continuing the unipolar field (21)

$$\frac{t}{\tau} \longrightarrow \frac{t}{\tau} + i\frac{\pi}{2}, \quad E_0 \longrightarrow iE_0. \tag{27}$$

The relativistic WKB action under the conformal mapping $z = e^{t/\tau}$ takes the form

$$\mathcal{S}_{C_J^{(1)}} = \frac{i}{2}\tau \oint_{C_J^{(1)}} \frac{dz}{z(z^2+1)} \Big[\sqrt{\mu^2(z^2+1)^2 + \left[k_\parallel(z^2+1) - 2qE_0\tau z\right]^2 \pm 2iqE_0 z(z^2-1)} + \sqrt{\mu^2(z^2+1)^2 + \left[k_\parallel(z^2+1) + 2qE_0\tau z\right]^2 \pm 2iqE_0 z(z^2-1)}\Big]. \tag{28}$$

Simple poles in the complex z-plane are $z = \infty$ and $z = \pm i$. The common factor comes from the residue contribution at $z = \infty$

$$\mathcal{S}_{\mathbf{k},\sigma=\pm\frac{1}{2},(\infty)} = 2\pi\tau\omega_{\mathbf{k}}(z=\infty). \tag{29}$$

The residue contributions for each spin component from the poles at $z = \pm i$ are

$$\mathcal{S}_{\mathbf{k},\sigma=\frac{1}{2},(\pm i)} = 2\pi i qE_0\tau^2 \pm i\pi,$$
$$\mathcal{S}_{\mathbf{k},\sigma=-\frac{1}{2},(\pm i)} = 2\pi i qE_0\tau^2 \pm i\pi. \tag{30}$$

The fermionic formula (4) yields

$$N_{\mathbf{k}}^{\rm sp} \approx \frac{1}{2} e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)} \left|2\left(1 - 2e^{-2\pi i qE_0\tau^2} + e^{-4\pi i qE_0\tau^2}\right)\right| = 4e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)} \sin^2\left(\pi qE_0\tau^2\right). \tag{31}$$

The first term comes from any loop not enclosing the poles $z = \pm i$ and the second term from two independent loops enclosing only one pole at $z = \pm i$. The last term comes from any loop enclosing both poles $z = \pm i$. Note that pairs are not produced when

$$qE_0\tau^2 = n, \quad n = 0, 1, \cdots. \tag{32}$$

The Stokes phenomenon of no pair production is a consequence of bipolar electric field and the existence of a finite pair of simple poles with pure imaginary residues as observed in Ref. [3]. In the formulation (4), the Stokes phenomenon is due to constructive or destructive interference of paths. A physical interpretation is that pairs produced as entangled states during $t = (-\infty, 0)$ exactly annihilate each other during the remaining time $t = (0, \infty)$.

V. Conclusion

In this paper we have elaborated and applied the phase-integral formulation to fermion production in electric fields of analytical functions in the complex time plane. The leading term, at least, of the fermion production is given by the formula (4). If the complex frequency of spin-1/2 fermion has pairs of finite simple poles as well as the simple pole from the infinity, all possible independent paths enclosing finite simple poles provide channels for pair production with their residues and another residue from the infinity pole determining the overall factor. In particular, pairs of poles with pure imaginary residues exhibit a rich structure of pair production such as the Stokes phenomenon due to constructive or destructive interferences of paths.

A time-antisymmetric bipolar electric field is shown to lead to the Stokes phenomenon of boson pair production while a unipolar electric field gives non-vanishing pair production in the future infinity [3]. It has been observed for boson pair production that (i) the existence of a pair of finite simple poles and (ii) the pure imaginary residues at the finite simple poles are the condition for the Stokes phenomena of constructive or destructive interference of pair production. The solitonic electric fields in Refs. [17,18] belong to this category and another class of bipolar electric field has been introduced [3].

We have studied the fermion production in the phase-integral formulation for the Sauter, the unipolar and bipolar electric fields in Ref. [3]. The Schwinger formulae for fermions show different functional dependence as summarized in Table 1. The time-antisymmetric bipolar electric field leads to the Stokes phenomena for pair production of bosons and fermions. An intriguing feature of the Stokes phenomena is the parameter condition for zero-pair production: Eq. (32) for fermions and for bosons

$$qE_0\tau^2 = n + \frac{1}{2}, \quad n = 0, 1, \cdots. \tag{33}$$

The natural number corresponds to the soliton numbers for the solitonic gauge fields for bosons [17,18]. This is reminiscent of the inversion of spin-statistics of bosons and fermions [19], but the physical reasoning for this requires a further study. Another interesting observation without an apparent explanation is that the fermion

Table 1. Comparison between boson and fermion production.

Fields \ Particles	Bosons [3]	Fermions
Unipolar electric field	$E_{\parallel}(t) = \frac{E_0 \cosh(t/\tau)}{\sinh^2(t/\tau)}$	$E_{\parallel}(t) = \frac{E_0 \cosh(t/\tau)}{\sinh^2(t/\tau)}$
$\mathcal{N}_{\mathbf{k}}$	$4e^{-2\pi\tau(\omega_{\mathbf{k}}(t=\infty)+qE_0\tau)} \cosh^2(\pi q E_0 \tau^2)$	$4e^{-2\pi\tau(\omega_{\mathbf{k}}(t=\infty)+qE_0\tau)} \sinh^2(\pi q E_0 \tau^2)$
Bipolar electric fields	$E_{\parallel}(t) = \frac{E_0 \sinh(t/\tau)}{\cosh^2(t/\tau)}$	$E_{\parallel}(t) = \frac{E_0 \sinh(t/\tau)}{\cosh^2(t/\tau)}$
$\mathcal{N}_{\mathbf{k}}$	$4e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)} \cos^2(\pi q E_0 \tau^2)$	$4e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)} \sin^2(\pi q E_0 \tau^2)$

(boson) production in the bipolar electric field is an analytical continuation of the boson (fermion) production in the unipolar electric field under $E_0 \leftrightarrows iE_0$ and vice versa.

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