Classes of graphs with no long cycle as a vertex-minor are polynomially \( \chi \)-bounded

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A class \( \mathcal{G} \) of graphs is \( \chi \)-bounded if there is a function \( f \) such that for every graph \( G \in \mathcal{G} \) and every induced subgraph \( H \) of \( G \), \( \chi(H) \leq f(\omega(H)) \). In addition, we say that \( \mathcal{G} \) is polynomially \( \chi \)-bounded if \( f \) can be taken as a polynomial function. We prove that for every integer \( n \geq 3 \), there exists a polynomial \( f \) such that \( \chi(G) \leq f(\omega(G)) \) for all graphs with no vertex-minor isomorphic to the cycle graph \( C_n \). To prove this, we show that if \( \mathcal{G} \) is polynomially \( \chi \)-bounded, then so is the closure of \( \mathcal{G} \) under taking the 1-join operation.

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1. Introduction

A class $\mathcal{G}$ of graphs is said to be hereditary if for every $G \in \mathcal{G}$, every graph isomorphic to an induced subgraph of $G$ belongs to $\mathcal{G}$. A class $\mathcal{G}$ of graphs is $\chi$-bounded if there is a function $f$ such that for every graph $G \in \mathcal{G}$ and every induced subgraph $H$ of $G$, $\chi(H) \leq f(\omega(H))$. The function $f$ is called a $\chi$-bounding function. This concept was first formulated by Gyárfás [9]. In particular, we say that $\mathcal{G}$ is polynomially $\chi$-bounded if $f$ can be taken as a polynomial function.

Recently, many open problems on $\chi$-boundedness have been resolved; see a recent survey by Scott and Seymour [14]. Yet we do not have much information on graph classes that are polynomially $\chi$-bounded. For instance, Gyárfás [9] showed that the class of $P_n$-free graphs is $\chi$-bounded but it is still open [8,13] whether it is polynomially $\chi$-bounded for $n \geq 5$. Regarding polynomially $\chi$-boundedness, Esperet proposed the following question, which remains open.

**Question 1.1** (Esperet; see [10]). Is every $\chi$-bounded class of graphs polynomially $\chi$-bounded?

Towards answering this question, it is interesting to know some graph operations that preserve the property of polynomial $\chi$-boundedness. If we have such graph operations, then we can use them to generate polynomially $\chi$-bounded graph classes.

In this direction, Chudnovsky, Penev, Scott, and Trotignon [4] showed that if a hereditary class $\mathcal{C}$ is polynomially $\chi$-bounded, then its closure under disjoint union and substitution is again polynomially $\chi$-bounded.

We prove the analog of their result for the 1-join. For graphs $G_1$ and $G_2$ with $|V(G_1)|, |V(G_2)| \geq 3$ and $V(G_1) \cap V(G_2) = \emptyset$, we say that a graph $G$ is obtained from $G_1$ and $G_2$ by 1-join if there are vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that $G$ is obtained from the disjoint union of $G_1$ and $G_2$ by deleting $v_1$ and $v_2$ and adding all edges between every neighbor of $v_1$ in $G_1$ and every neighbor of $v_2$ in $G_2$. If so, then we say that $G$ is the 1-join of $(G_1, v_1)$ and $(G_2, v_2)$. For a class $\mathcal{G}$, let $\mathcal{G}^k$ be its closure under disjoint union and 1-join. Note that if $\mathcal{G}$ is closed under taking isomorphisms, then so is $\mathcal{G}^k$. We will see in Section 2 that $\mathcal{G}^k$ is hereditary if $\mathcal{G}$ is hereditary.

**Theorem 1.2.** If $\mathcal{G}$ is a polynomially $\chi$-bounded class of graphs, then so is $\mathcal{G}^k$.

Dvořák and Král [7] and Kim [11] independently showed that for every hereditary class $\mathcal{G}$ of graphs that is $\chi$-bounded, its closure under taking the 1-joins is again $\chi$-bounded. However, in both papers, the $\chi$-bounding function $g$ for the new class is recursively defined as $g(n) = O(f(n)g(n-1))$ for a $\chi$-bounding function $f$ for $\mathcal{G}$. So, $g(n)$ is exponential under their constructions.

We shall see that if $f$ is a polynomial, then $g(n-1)$ in the recurrence relation can be replaced by some polynomial $f^*$. This technique allows us to prove Theorem 1.2.
As an application, we investigate the following conjecture of Geelen proposed in 2009. The definition of vertex-minors will be reviewed in Section 2.

**Conjecture 1.3** (Geelen; see [7]). For every graph $H$, the class of graphs with no vertex-minor isomorphic to $H$ is $\chi$-bounded.

Conjecture 1.3 is known to be true when $H$ is a wheel graph, shown by Choi, Kwon, Oum, and Wollan [1]. Motivated by the question of Esperet, we may ask the following.

**Question 1.4.** Is it true that for every graph $H$, the class of graphs with no vertex-minor isomorphic to $H$ is polynomially $\chi$-bounded?

If this holds for $H$, then the class of $H$-vertex-minor free graphs satisfies the *Erdős-Hajnal property*, which means that there is a constant $c > 0$ such that every graph $G$ in this class has an independent set or a clique of size at least $|V(G)|^c$. Recently, Chudnovsky and Oum [3] proved that the Erdős-Hajnal property holds for the class of $H$-vertex-minor free graphs for all $H$.

We write $P_n$ to denote the path graph on $n$ vertices and $C_n$ to denote the cycle graph on $n$ vertices. Let $K_n \boxtimes K_n$ be the graph on $2n$ vertices $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ such that $\{a_1, a_2, \ldots, a_n\}$ is a clique, $\{b_1, b_2, \ldots, b_n\}$ is a stable set, and for all $1 \leq i, j \leq n$, $a_i$ is adjacent to $b_j$ if and only if $i \geq j$. See Fig. 1 for an illustration of $K_6 \boxtimes K_6$.

In Section 4, we prove the following theorem.

**Theorem 1.5.** Let $n \geq 4$. If a graph $G$ has no induced subgraph isomorphic to $P_n$ or $K_{[n/2]} \boxtimes K_{[n/2]}$, then

$$\chi(G) \leq (n - 3)^{[n/2]-1}\omega(G)^{[n/2]-1}.$$  

We use this graph $K_{[n/2]} \boxtimes K_{[n/2]}$ because it has a vertex-minor isomorphic to $P_n$, shown by Kwon and Oum [12, Lemma 2.8].

**Lemma 1.6** (Kwon and Oum [12, Lemma 2.8]). The graph $K_{[n/2]} \boxtimes K_{[n/2]}$ has a vertex-minor isomorphic to $P_n$.

This allows us to obtain the following corollary of Theorem 1.5.
Corollary 1.7. The class of graphs with no vertex-minor isomorphic to $P_n$ is polynomially $\chi$-bounded.

Kwon and Oum [12] proved the following theorem, stating that a prime graph with a long induced path must contain a long induced cycle as a vertex-minor. A graph is prime if it is not the 1-join of $(G_1,v_1)$ and $(G_2,v_2)$ for some graphs $G_1$, $G_2$ with $|V(G_1)|, |V(G_2)| \geq 3$.

Theorem 1.8 (Kwon and Oum [12]). If a prime graph has an induced path of length $[6.75n^7]$, then it has a cycle of length $n$ as a vertex-minor.

We deduce the following stronger theorem from Corollary 1.7 by using Theorems 1.2 and 1.8. This answers Question 1.4 for a long cycle.

Theorem 1.9. The class of graphs with no vertex-minor isomorphic to $C_n$ is polynomially $\chi$-bounded.

Proof. Let $\mathcal{G}$ be the class of graphs having no vertex-minor isomorphic to $P_m$ for $m = [6.75n^7]$. By Corollary 1.7, $\mathcal{G}$ is polynomially $\chi$-bounded. By Theorem 1.2, $\mathcal{G}^k$ is polynomially $\chi$-bounded. Let $\mathcal{H}$ be the class of graphs having no vertex-minor isomorphic to $C_n$. Let $G \in \mathcal{H}$. We claim that $G \in \mathcal{G}^k$. We may assume that $G$ is connected. Every connected prime induced subgraph of $G$ is in $\mathcal{G}$ by Theorem 1.8. Then $G$ can be obtained from connected prime induced subgraphs of $G$ by taking 1-join repeatedly. Thus, $G \in \mathcal{G}^k$. This proves that $\mathcal{H} \subseteq \mathcal{G}^k$. \qed

Because $C_m$ contains $C_n$ as a vertex-minor whenever $m \geq n$, we may ask a stronger question on whether or not the class of graphs with no induced subgraph isomorphic to $C_m$ for some $m \geq n$ is polynomially $\chi$-bounded. It is not known. The following theorem of Chudnovsky, Scott, and Seymour [6] was initially a conjecture of Gyárfás [9] in 1985.

Theorem 1.10 (Chudnovsky, Scott, and Seymour [6]). The class of graphs with no induced subgraph isomorphic to a graph in $\{C_m : m \geq n\}$ is $\chi$-bounded.

We remark that as far as we know, it is not known whether the class of graphs with no $P_5$ induced subgraph is polynomially $\chi$-bounded.

This paper is organized as follows. We will review necessary definitions in Section 2. In Section 3, we will present a proof of Theorem 1.2. In Section 4, we will prove Theorem 1.5. In Section 5, we discuss related problems.
2. Preliminaries

All graphs in this paper are simple and undirected. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. A clique of a graph is a set of pairwise adjacent vertices. For a graph $G$, let $\omega(G)$ be the maximum number of vertices in a clique of $G$ and $\chi(G)$ be the chromatic number of $G$.

Let $G$ be a graph. For a vertex subset $S$ of $G$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. For a vertex $v$ of $G$, we denote by $G \setminus v$ the graph obtained from $G$ by removing $v$. For an edge $e$ of $G$, we write $G \setminus e$ to denote the subgraph obtained from $G$ by deleting $e$. For $v \in V(G)$, let $N_G(v)$ be the set of neighbors of $v$ in $G$. For a set $X$ of vertices, let $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$.

For two graphs $G_1$ and $G_2$, the disjoint union of $G_1$ and $G_2$ is a graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ where $G_1'$ is an isomorphic copy of $G_1$ and $G_2'$ is an isomorphic copy of $G_2$ such that $V(G_1') \cap V(G_2') = \emptyset$. If $V(G_1') \cap V(G_2') = \emptyset$, then we take $G_1' = G_1$ and $G_2' = G_2$ for convenience.

For two graphs $G_1$ and $G_2$ on disjoint vertex sets and a vertex $v \in V(G_1)$, we say that a graph $G$ is obtained from $G_1$ by substituting $G_2$ for $v$ in $G_1$, if

- $V(G) = (V(G_1) \setminus \{v\}) \cup V(G_2)$,
- $E(G) = E(G_1 \setminus v) \cup E(G_2) \cup \{xy : x \in N_{G_1}(v), y \in V(G_2)\}$.

For two sets $A$, $A'$ with $A \subseteq A'$, we say that a function $f' : A' \to B$ extends a function $f : A \to B$ if $f'(a) = f(a)$ for all $a \in A$.

Lemma 2.1. If $G$ is a hereditary class of graphs, then so is $G^{k}$.

Proof. We show that for every $G \in G^{k}$ and $v \in V(G)$, $G \setminus v \in G^{k}$. We proceed by induction on $|V(G)|$. If $G \in G$, then we are done since $G$ is hereditary.

So, we may assume that $G \notin G$. Then, $G$ is the disjoint union of $G_1$ and $G_2$ for some $G_1, G_2 \in G^{k}$ or the 1-join of $(G_1, v_1)$ and $(G_2, v_2)$ for some $G_1, G_2 \in G^{k}$ and $v_i \in V(G_i)$ for $i = 1, 2$ where $|V(G_1)|, |V(G_2)| \geq 3$.

In the first case, we may assume that $v \in V(G_1)$. Then, by the induction hypothesis, $G_1 \setminus v \in G^{k}$, and since $G \setminus v$ is the disjoint union of $G_1 \setminus v$ and $G_2$, it follows that $G \setminus v$ is contained in $G^{k}$.

In the second case, by symmetry, we may assume that $v \in V(G_1) \setminus \{v_1\}$. By the induction hypothesis, $G_1 \setminus v \in G^{k}$. If $|V(G_1)| > 3$, then $G \setminus v$ is in $G^{k}$ since it is the 1-join of $(G_1 \setminus v, v_1)$ and $(G_2, v_2)$. If $|V(G_1)| = 3$, then $G \setminus v$ is isomorphic to either $G_2$ or the disjoint union of $K_1$ and $G_2 \setminus v_2$. In either case $G \setminus v$ is contained in $G^{k}$. □

Vertex-minors Now we are going to define vertex-minors. Actually, the readers may skip this part, if they assume Lemma 1.6 and Theorem 1.8 from other papers.
For a vertex $v$ in a graph $G$, the local complementation at $v$ results in the graph obtained from $G$ by replacing the subgraph of $G$ induced on $N_G(v)$ by its complement. We write $G \ast v$ to denote the graph obtained from $G$ by applying local complementation at $v$. In other words, $G \ast v$ is a graph on $V(G)$ such that two distinct vertices $x$, $y$ are adjacent in $G \ast v$ if and only if exactly one of the following holds.

(i) Both $x$ and $y$ are neighbors of $v$ in $G$.
(ii) $x$ is adjacent to $y$ in $G$.

A graph $H$ is locally equivalent to $G$ if $H$ can be obtained from $G$ by a sequence of local complementations. We say that a graph $H$ is a vertex-minor of a graph $G$ if $H$ is an induced subgraph of a graph locally equivalent to $G$.

For an edge $uv$ of a graph $G$, the pivot at $uv$ is an operation to obtain $G \ast v \ast u \ast v$ from $G$. We write $G \wedge uv := G \ast u \ast v \ast u$. We say that a graph $H$ is a pivot-minor of a graph $G$ if $H$ is obtained from $G$ by a sequence of pivots and vertex deletions.

3. Polynomials $\chi$-boundedness for 1-join

For a class $\mathcal{G}$ of graphs, let $\mathcal{G}^\ast$ be the closure of $\mathcal{G}$ under disjoint union and substitution. We will use the following result due to Chudnovsky, Penev, Scott, and Trotignon [4].

**Theorem 3.1** *(Chudnovsky, Penev, Scott, and Trotignon [4])* If $\mathcal{G}$ is a polynomially $\chi$-bounded class of graphs, then $\mathcal{G}^\ast$ is polynomially $\chi$-bounded.

The following observation relates two graph classes $\mathcal{G}^k$ and $\mathcal{G}^\ast$.

**Lemma 3.2.** Let $\mathcal{G}$ be a hereditary class of graphs. If $G \in \mathcal{G}^k$ and $v \in V(G)$, then $G[N_G(v)] \in \mathcal{G}^\ast$.

**Proof.** We prove by induction on $|V(G)|$.

If $G \in \mathcal{G}$, then we are done, since $\mathcal{G}$ is closed under induced subgraphs and $\mathcal{G} \subseteq \mathcal{G}^\ast$. If $G$ is the disjoint union of two graphs $G_1, G_2$ from $\mathcal{G}^k$ and $v \in V(G_1)$, then by the induction hypothesis on the graph $G_1$, the claim follows.

Suppose that $G$ is the 1-join of two graphs $(G_1, v_1)$ and $(G_2, v_2)$ where $G_1, G_2 \in \mathcal{G}^k$ and $|V(G_1)|, |V(G_2)| \geq 3$. Without loss of generality, we may assume that $v \in V(G_1 \setminus v_1)$. Let $G_1[N_{G_1}(v)] = G_v$, and $G_2[N_{G_2}(v_2)] = G_2$. Then, by the induction hypothesis, $G_v \in \mathcal{G}^\ast$, and $G_2' \in \mathcal{G}^\ast$ because $|V(G_1)|, |V(G_2)| < |V(G)|$.

Since $\mathcal{G}$ is hereditary, so is $\mathcal{G}^\ast$. We may assume that $G_2'$ has at least one vertex because otherwise $G[N_{G_2}(v)] = G_v \setminus v_1 \in \mathcal{G}^\ast$. We may also assume that $v$ is adjacent to $v_1$ in $G_1$ because otherwise $G[N_G(v)] = G_v \in \mathcal{G}^\ast$. Then $G[N_G(v)]$ can be obtained from $G_v$ by substituting $G_2'$ for $v_1$ and therefore $G[N_G(v)]$ belongs to $\mathcal{G}^\ast$. This completes the proof. □
Let us now define a structure to describe how a connected graph in $\mathcal{G}^k$ is composed from graphs in $\mathcal{G}$. A composition tree is a triple $(T, \phi, \psi)$ of a tree $T$, a map $\phi$ defined on $V(T)$ and a map $\psi$ defined on $E(T)$ such that

- for $t \in V(T)$, $\phi(t)$ is a connected graph, say $G_t$, on at least 3 vertices where graphs in $\{G_t : t \in V(T)\}$ are vertex-disjoint,
- for $st \in E(T)$, $\psi(st) = \{u, v\}$ for some $u \in V(G_s)$ and $v \in V(G_t)$, and
- for distinct $e_1 \neq e_2 \in E(T)$, $\psi(e_1)$ and $\psi(e_2)$ are disjoint.

If a composition tree $(T, \phi, \psi)$ is given, then one can construct a connected graph $G$ from $(T, \phi, \psi)$ by taking 1-joins repeatedly as follows:

- if $|V(T)| = 1$, say $V(T) = \{t\}$, then $G = \phi(t)$,
- if $|V(T)| > 1$, let $e = t_1t_2 \in E(T)$ and $T_i$ be the subtree of $T \setminus e$ containing $t_i$ for each $i = 1, 2$. Let $\phi_i$ be the restriction of $\phi$ on $V(T_i)$ and $\psi_i$ be the restriction of $\psi$ on $E(T_i)$ for each $i = 1, 2$. Let $G_i$ be a graph constructed from $(T_i, \phi_i, \psi_i)$ for $i = 1, 2$. Then, $G$ is the 1-join of $(G_1, v_1)$ and $(G_2, v_2)$ where $v_i \in V(G_i) \cap \psi(e)$. It is straightforward to see that the choice of $e$ does not make any difference to the obtained graph $G$.

If a vertex $v$ of $\phi(t)$ for some node $t$ of $T$ is in $\psi(e)$ for some edge $e$ of $T$, then $v$ is called a marker vertex. After applying all 1-joins, marker vertices will disappear.

**Lemma 3.3.** Let $\mathcal{G}$ be a class of graphs. Let $G$ be a connected graph in $\mathcal{G}^k$ with at least three vertices. Then there exists a composition tree $(T, \phi, \psi)$ that constructs $G$ such that $\phi(t) \in \mathcal{G}$ for every node $t$ of $T$.

**Proof.** We proceed by induction on $|V(G)|$. We may assume that $G \notin \mathcal{G}$. Since $G$ is connected, $G$ is the 1-join of $(G_1, v_1)$ and $(G_2, v_2)$ for some graphs $G_1$, $G_2$ in $\mathcal{G}^k$ and $v_1 \in V(G_1)$, $v_2 \in V(G_2)$ where $|V(G_1)|, |V(G_2)| \geq 3$. Since $G$ is connected, both $G_1$ and $G_2$ are connected. By the induction hypothesis, we obtain two composition trees. We can combine them to obtain a composition tree $(T, \phi, \psi)$ constructing $G$. 

**Lemma 3.4.** Let $c_1$, $c_2$ be positive integers. Let $G$ be a connected graph constructed by a composition tree $(T, \phi, \psi)$ such that $\phi(t)$ is $c_1$-colorable for each node $t$ of $T$ and $G[N_G(w)]$ is $c_2$-colorable for each vertex $w$ of $G$. Let $v$ be a vertex of $G$. Then for every proper $c_2$-coloring $\beta$ of $G[N_G(v)]$, there exist functions $\alpha' : V(G) \setminus \{v\} \to \{0, 1, 2, \ldots, c_1\}$ and $\beta' : V(G) \setminus \{v\} \to \{1, 2, \ldots, c_2\}$ such that

1. $\alpha'(w) = 0$ and $\beta'(w) = \beta(w)$ for every neighbor $w$ of $v$,
2. $c = \alpha' \times \beta'$ is a proper $(c_1 + 1)c_2$-coloring of $G \setminus v$. 

Proof. We proceed by induction on $|V(G)|$.

If $|V(T)| = 1$, then $G = \phi(t)$ for the unique node $t$ of $T$ and so $G$ has a proper $c_1$-coloring $h : V(G \setminus v) \to \{1, 2, \ldots, c_1\}$. We define $\alpha'$ and $\beta'$ on $V(G \setminus v)$ as follows:

$$
\alpha'(w) = \begin{cases} 
0 & \text{if } w \in N_G(v), \\
h(w) & \text{otherwise,}
\end{cases}
\quad \text{and} \quad
\beta'(w) = \begin{cases} 
\beta(w) & \text{if } w \in N_G(v), \\
1 & \text{otherwise.}
\end{cases}
$$

Clearly, $\alpha' \times \beta'$ is a proper $(c_1 + 1)c_2$-coloring of $G \setminus v$.

Thus we may assume $|V(T)| > 1$. Let $t_0$ be the unique node of $T$ such that $v \in V(\phi(t_0))$. Let $G_0 := \phi(t_0)$. Let $t_1, t_2, \ldots, t_m$ be the neighbors of $t_0$ in $T$. For each $i \in \{1, 2, \ldots, m\}$, let $v_i \in V(G_0)$ and $u_i \in V(\phi(t_i))$ be vertices such that $\psi(t_0 t_i) = \{v_i, u_i\}$ and let $T_i$ be the connected component of $T \setminus t_0$ containing $t_i$. For each $i \in \{1, 2, \ldots, m\}$, let $\phi_i$ be the restriction of $\phi$ on $V(T_i)$ and $\psi_i$ be the restriction of $\psi$ on $E(T_i)$ and $G_i$ be the graph constructed from a composition tree $(T_i, \phi_i, \psi_i)$.

Let $h : V(G_0) \to \{1, 2, \ldots, c_1\}$ be a proper $c_1$-coloring of $G_0$. Let $\alpha'_0, \beta'_0$ be maps defined on $V(G_0) \setminus \{v, v_1, \ldots, v_m\}$ such that for $w \in V(G_0) \setminus \{v, v_1, \ldots, v_m\}$,

$$
\alpha'_0(w) = \begin{cases} 
0 & \text{if } w \notin N_{G_0}(v), \\
h(w) & \text{otherwise,}
\end{cases}
\quad \text{and} \quad
\beta'_0(w) = \begin{cases} 
\beta(w) & \text{if } w \notin N_{G_0}(v), \\
1 & \text{otherwise.}
\end{cases}
$$

Now we are going to define, for each $i \in \{1, 2, \ldots, m\}$, a proper $c_2$-coloring $\beta_i$ of $G_i[N_{G_i}(u_i)]$. If $v_i$ is adjacent to $v$ in $G_0$, then $N_{G_i}(u_i)$ is a subset of $N_G(v)$ and so let us define $\beta_i$ to be the proper $c_2$-coloring of $G[N_{G_i}(u_i)]$ induced by $\beta$.

If $v_i$ is non-adjacent to $v$ in $G_0$, then we claim that there exists a vertex $y$ of $G$ such that $N_{G_i}(u_i) \subseteq N_G(y)$. Since $G_0$ is connected, $v_i$ has a neighbor $x$ in $G_0$. If $x$ is not a marker vertex, then $N_{G_i}(u_i) \subseteq N_G(x)$. If $x$ is a marker vertex, say $x = v_j$ for some $j \neq i$, then there exists a neighbor $y$ of $u_j$ in $G_j$ because $G_j$ is connected and $|V(G_j)| \neq 1$. Now we observe that $N_{G_i}(u_i) \subseteq N_G(y)$. This proves the claim. By the claim, we can define $\beta_i$ as a proper $c_2$-coloring of $G_i[N_{G_i}(u_i)]$ induced by a proper $c_2$-coloring of $G[N_{G_i}(y)]$.

Observe that $|V(G_i)| < |V(G)|$ for all $i \in \{1, 2, \ldots, m\}$ because $G_0$ has at least three vertices. Now, by the induction hypothesis, for each $i \in \{1, 2, \ldots, m\}$, there exist maps $\alpha'_i : V(G_i \setminus u_i) \to \{0, 1, \ldots, c_1\}$ and $\beta'_i : V(G_i \setminus u_i) \to \{1, 2, \ldots, c_2\}$ such that for every $w \in N_{G_i}(u_i)$,

$$
\alpha'_i(w) = \begin{cases} 
h(v_i) & \text{if } v_i \text{ is non-adjacent to } v \text{ in } G_0, \\
0 & \text{otherwise,}
\end{cases}
\quad \text{and} \quad
\beta'_i(w) = \beta_i(w),
$$

and $\alpha'_i \times \beta'_i$ is a proper $(c_1 + 1)c_2$-coloring of $G_i \setminus u_i$, because we may swap colors 0 and $h(v_i)$ in $\alpha'_i$ after applying the induction hypothesis.
Now we define maps $\alpha'$ and $\beta'$ on $V(G) \setminus \{v\}$ such that for $w \in V(G) \setminus \{v\}$,

$$
\alpha'(w) = \alpha'_i(w) \text{ and } \beta'(w) = \beta'_i(w) \text{ if } w \in V(G_i) \text{ for some } i \in \{0, 1, 2, \ldots, m\}.
$$

Clearly, $\beta'$ extends $\beta$. In addition, $\alpha'(w) = 0$ for all neighbors $w$ of $v$ in $G$.

We claim that $c = \alpha' \times \beta'$ is a proper coloring of $G \setminus v$. Let $x, y \in V(G \setminus v)$ be adjacent vertices in $G \setminus v$. If both $x$ and $y$ are neighbors of $v$, then $\beta'(x) = \beta(x) \neq \beta(y) = \beta'(y)$.

So we may assume that $y$ is not a neighbor of $v$.

- If $x, y \in V(G_0)$, then $\alpha'(x) \neq \alpha'(y)$ because $\alpha'(x) \in \{0, h(x)\}$ and $\alpha'(y) = h(y) \neq 0$.
- If $x, y \in V(G_i)$ for some $i \in \{1, 2, \ldots, m\}$, then $(\alpha'(x), \beta'(x)) \neq (\alpha'(y), \beta'(y))$ because $\alpha'_i \times \beta'_i$ is a proper coloring of $G_i \setminus u_i$.
- If $V(G_0)$ contains exactly one of $x$ and $y$, say $x$, then there exists $i \in \{1, 2, \ldots, m\}$ such that $y \in V(G_i)$. Then $x$ is adjacent to $v_i$ in $G_0$ and $y$ is adjacent to $u_i$ in $G_i$. Since $y$ is not adjacent to $v$, $v_i$ is not adjacent to $v$ in $G_0$ and so $\alpha'(y) = \alpha'_i(y) = h(v_i)$. As $\alpha'(x) \in \{0, h(x)\}$, we deduce that $\alpha'(x) \neq \alpha'(y)$.
- If $x \in V(G_i)$ and $y \in V(G_j)$ for distinct $i, j \in \{1, 2, \ldots, m\}$, then $x$ is adjacent to $u_i$ in $G_i$, $v_i$ is adjacent to $v_j$ in $G_0$, and $u_j$ is adjacent to $y$ in $G_j$. Since $y$ is not adjacent to $v$, $v_j$ is not adjacent to $v$ in $G_0$ and so $\alpha'(y) = \alpha'_j(y) = h(v_j)$. Note that $\alpha'(x) \in \{0, h(v_i)\}$ and therefore $\alpha(x) \neq \alpha(y)$ because $h$ is a proper coloring of $G_0$.

Therefore, $c$ is a proper coloring of $G \setminus v$. This completes the proof. □

**Proof of Theorem 1.2.** We may assume that $\mathcal{G}$ is hereditary, by replacing $\mathcal{G}$ with the closure of $\mathcal{G}$ under isomorphism and taking induced subgraphs, if necessary.

Let $f$ be a $\chi$-bounding function for $\mathcal{G}$ that is a polynomial. We may assume that $1 \leq f(0) \leq f(1) \leq f(2) \leq \cdots$, by replacing $f(x) = \sum_i a_i x^i$ with $\sum_i |a_i| x^i$ if needed. By Theorem 3.1, $\mathcal{G}^*$ is $\chi$-bounded by a polynomial $f^*$. We may assume that $1 \leq f^*(0) \leq f^*(1) \leq f^*(2) \leq \cdots$.

We claim that

$$
\forall G \in \mathcal{G}^k. \chi(G) \leq (f(\omega(G)) + 1)f^*(\omega(G) - 1)
$$

for all $G \in \mathcal{G}^k$. This claim implies the theorem because $\mathcal{G}^k$ is hereditary by Lemma 2.1.

Let $k = \omega(G)$. We may assume that $k > 1$. We may assume that $G$ is connected because $\mathcal{G}^k$ is hereditary and both $f$ and $f^*$ are non-decreasing. We may assume that $G$ has at least three vertices. By Lemma 3.3, $G$ has a composition tree $(T, \phi, \psi)$ with $\phi(x) \in \mathcal{G}$ for every node $x$ of $T$. Note that $\omega(\phi(x)) \leq k$ because $\phi(x)$ is isomorphic to an induced subgraph of $G$ and therefore $\chi(\phi(x)) \leq f(\omega(\phi(x))) \leq f(k)$. For each vertex $w \in V(G)$, $\omega(G[N_G(w)]) \leq k - 1$ and $G[N_G(w)]$ belongs to $\mathcal{G}^*$ by Lemma 3.2, and so $G[N_G(w)]$ is $f^*(k - 1)$-colorable. Let $v$ be a vertex of $G$. By Lemma 3.4, there exists a proper $(f(k) + 1)f^*(k - 1)$-coloring $c = \alpha' \times \beta'$ of $G \setminus v$ such that $\alpha'(w) = 0$ for every
neighbor $w$ of $v$. Then we can easily extend this to a proper $(f(k) + 1)f^*(k - 1)$-coloring of $G$ by taking $\alpha'(v) \neq 0$. □

We remark that the same method can also prove that if $G$ is a class of graphs having an exponential $\chi$-bounding function, then so is $G^k$. This is because Chudnovsky, Penev, Scott, and Trotignon [4] also prove an analogue of Theorem 3.1 for classes of graphs having exponential $\chi$-bounding functions.

4. Graphs without $P_n$ or $K_{[n/2]} \square \overline{K_{[n/2]}}$ induced subgraphs

Now we are ready to prove Theorem 1.5, which states that the class of graphs with no induced subgraph isomorphic to $P_n$ or $K_{[n/2]} \square \overline{K_{[n/2]}}$ is polynomially $\chi$-bounded.

For this section, a path from $v$ to $w$ is a sequence $v_0v_1\cdots v_\ell$ of distinct vertices such that $v_0 = v$, $v_\ell = w$, and $v_i$ is adjacent to $v_{i-1}$ for all $i = 1, 2, \ldots, \ell$. We say a path $Q := w_0w_1\cdots w_m$ extends a path $P := v_0v_1\cdots v_\ell$ if $m \geq \ell$ and $v_i = w_i$ for all $i \in \{0, 1, \ldots, \ell\}$.

For an induced path $P$ from $v$ to $w$ in $G$, we write $\Omega(G, P)$ to denote $G \setminus (V(P) \cup N_G(V(P \setminus w)))$. A component of $\Omega(G, P)$ is attached to $P$ if it contains a neighbor of $w$. A component $C$ of $\Omega(G, P)$ is $d$-good if the neighbors of $w$ in $C$ induce a graph of chromatic number larger than $d$. We say $C$ is $d$-bad if it is not $d$-good. We say $P$ is $d$-good in $G$ if $\Omega(G, P)$ has a $d$-good component.

**Lemma 4.1.** Let $k, d$ be positive integers. Let $G$ be a graph. If $\omega(G) \leq k$ and $\chi(G) > kd$, then $G$ has a path $P$ of length 1 such that $\Omega(G, P)$ has a component $C$ attached to $P$ and $\chi(C) > d$.

**Proof.** We may assume that $G$ is connected. Let $K$ be a maximum clique of $G$. By assumption, $|K| \leq k$. For each vertex $x$ of $K$, let $H_x = G \setminus N_G(x)$. Since $K$ is a maximum clique, for every vertex $y$, there is $x \in K$ such that $y \in V(H_x)$ and therefore $\chi(G) \leq \sum_{x \in K} \chi(H_x)$. So there exists $x \in K$ such that $\chi(H_x) > d$. Let $C'$ be a component of $H_x$ such that $\chi(C') = \chi(H_x)$. Since $\chi(C') > 1$ and $x$ is an isolated vertex in $H_x$, $x \notin V(C')$. Since $G$ is connected, there is a shortest path $v_0v_1\cdots v_\ell$ from $x = v_0$ to some vertex $v_\ell$ of $C'$. Let $v := v_{\ell-2}$, $w := v_{\ell-1}$, and $P := vw$. Let $C$ be the component of $G \setminus N_G(v)$ containing $C'$. Then $C$ is a component attached to $P$ and $\chi(C) > d$. □

**Lemma 4.2.** If a graph $G$ has an induced path $P$ of length at least 1 and $\Omega(G, P)$ has a $d$-bad component $C$ attached to $P$ with $\chi(C) > d$, then there exist an induced path $P'$ extending $P$ by exactly 1 edge and a component $C'$ of $\Omega(G, P')$ attached to $P'$ such that

$$\chi(C') \geq \chi(C) - d.$$

**Proof.** Let $w$ be the last vertex of $P$. Let $C_w$ be the subgraph of $C$ induced by the neighbors of $w$. Since $C$ is $d$-bad, $\chi(C_w) \leq d$ and therefore $\chi(C \setminus N_G(w)) \geq \chi(C) - \chi(C_w) \geq \chi(C) - d > 0$. So $C \setminus N_G(w)$ has a component $C'$ with $\chi(C') \geq \chi(C) - d$. Since
$C$ is connected, there is a vertex $w' \in V(C_w)$ adjacent to some vertex in $C'$. We obtain $P'$ by adding $w'$ as a last vertex to $P$. Then $C'$ is a component of $\Omega(G, P')$ attached to $P'$. □

**Lemma 4.3.** Let $n \geq 4$. Let $G$ be a graph having no induced subgraph isomorphic to $P_n$. Let $P$ be a path of length 1. If $\Omega(G, P)$ has a component $C$ attached to $P$ with $\chi(C) > d(n - 3)$, then $G$ has a $d$-good induced path $P'$ extending $P$.

**Proof.** Suppose that $G$ has no $d$-good induced path extending $P$. By applying Lemma 4.2 $(n - 3)$ times, we can find an induced path $P'$ of length $n - 2$ extending $P$ and a component $C'$ of $\Omega(G, P')$ attached to $P'$ such that $\chi(C') \geq \chi(C) - d(n - 3) > 0$. We obtain an induced path of length $n - 1$ by taking $P'$ and one vertex in $C'$ adjacent to the last vertex of $P'$. This contradicts the assumption that $G$ has no induced path on $n$ vertices. □

Now we are ready to prove Theorem 1.5.

**Theorem 1.5.** Let $n \geq 4$. If a graph $G$ has no induced subgraph isomorphic to $P_n$ or $K_{[n/2]} \boxtimes K_{[n/2]}$, then

$$
\chi(G) \leq (n - 3)^{\lceil n/2 \rceil - 1}\omega(G)^{\lceil n/2 \rceil - 1}.
$$

**Proof.** Let $k = \omega(G)$. We may assume that $G$ is connected. Let $G_0 = G$. Let $d_i = (n - 3)^{\lceil n/2 \rceil - i - 1}k^{\lceil n/2 \rceil - i - 1}$. Note that $\chi(G_0) > d_0$.

Inductively we will find, in $G_{i-1}$ of $\chi(G_{i-1}) > d_{i-1}$, an induced path $Q_i$ and connected induced subgraphs $C_i, G_i$ of $\chi(G_i) > d_i$ as follows. For $i = 1, \ldots, \lceil n/2 \rceil - 1$, by Lemmas 4.1 and 4.3, $G_{i-1}$ has a $d_i$-good induced path $Q_i$ of length at least 1, because $d_{i-1} = d_ik(n-3)$. Let $C_i$ be a $d_i$-good component of $\Omega(G_{i-1}, Q_i)$ attached to $Q_i$. Among all components of the subgraph of $C_i$ induced by the neighbors of the last vertex of $Q_i$, we choose a component $G_i$ of the maximum chromatic number. By the definition of a $d_i$-good component, $\chi(G_i) > d_i$. This constructs $G_1, G_2, \ldots, G_{\lceil n/2 \rceil - 1}$.

As $\chi(G_{\lceil n/2 \rceil - 1}) > d_{\lceil n/2 \rceil - 1} = 1$, $G_{\lceil n/2 \rceil - 1}$ contains at least one edge $xy$. By collecting the last two vertices of $Q_1, Q_2, \ldots, Q_{\lceil n/2 \rceil - 1}$ and $x, y$, we obtain an induced subgraph isomorphic to $K_{[n/2]} \boxtimes K_{[n/2]}$, contradicting the assumption on $G$. □

5. Discussions

Corollary 1.7 and Theorem 1.5 show that

$$
\chi(G) \leq (n - 3)^{\lceil n/2 \rceil - 1}\omega(G)^{\lceil n/2 \rceil - 1},
$$

for a graph $G$ with no vertex-minor isomorphic to $P_n$. It is natural to ask the following question.
Question 5.1. Do there exist a constant $c$ and a function $f(n)$ such that

$$\chi(G) \leq f(n)\omega(G)^c$$

for all integers $n$ and all graphs $G$ with no vertex-minor isomorphic to $P_n$?

If we replace the vertex-minor condition of $G$ in Question 5.1 with the condition having no induced subgraph isomorphic to $P_n$ or $K_{[n/2]} \nabla \overline{K}_{[n/2]}$, then the answer is no, by an argument in [14]. We include its proof for the completeness.

Proposition 5.2. For every constant $c$ and a function $f(n)$, there exist a graph $G$ and an integer $n$ such that $\chi(G) > f(n)\omega(G)^c$ and $G$ has no induced subgraph isomorphic to $P_n$ or $K_{[n/2]} \nabla \overline{K}_{[n/2]}$.

Proof. There are known lower bounds on Ramsey numbers; for instance, Spencer [15] showed that for fixed $k$, $R(k, t) > t^{(k-1)/2+o(1)}$ for all sufficiently large $t$. It implies that for fixed $k$, for all sufficiently large $t$, there exists a graph $G_t^{(k)}$ having no stable set of size $k$ and $\omega(G_t^{(k)}) < t$, $|V(G_t^{(k)})| > t^{(k-1)/2+o(1)}$. Then $\chi(G_t^{(k)}) \geq t^{(k-1)/2+o(1)}/k$.

Fix an integer $k$ so that $(k-1)/2 > c+1$. Let $c' := f(2k)$. Then for all sufficiently large $t$, $\chi(G_t^{(k)}) > t^{(k-1)/2+o(1)}/k \geq c't^c$. Then for all sufficiently large $t$, $\chi(G_t^{(k)}) > c'\omega(G)^c$ and $G_t^{(k)}$ has no induced subgraph isomorphic to $P_{2k}$ or $K_k \nabla \overline{K}_k$. $\square$

The bound (1) cannot be improved for $n = 4$, as $\chi(G) \geq \omega(G)$ in general. We will present the best possible bound for $n = 5$.

Theorem 5.3. If a graph $G$ has no vertex-minor isomorphic to $P_5$, then

$$\chi(G) \leq \omega(G) + 1.$$ 

The following proposition trivially implies Theorem 5.3. We denote by $W_n$ the wheel graph on $n + 1$ vertices.

Proposition 5.4. Every graph with no vertex-minor isomorphic to $P_5$ is perfect, unless it has a component isomorphic to $C_5$ or $W_5$.

In order to prove Proposition 5.4, we need to define the following graph classes. See Fig. 2 for an illustration.

- $W'_4$: the graph obtained from $W_4$ by deleting a spoke.
- Banner: the graph obtained from $C_4$ by adding a pendant edge.
- Bull: the graph obtained from $C_3$ by adding two pendant edges to distinct vertices of $C_3$. 

• Dart: the graph obtained from $K_4 \setminus e$ for some edge $e$ of $K_4$ by adding a pendant edge to a vertex of degree 3.

• HVN: the graph obtained from $K_4$ by adding a vertex of degree 2.

• Kite: the graph obtained from $K_4 \setminus e$ for some edge $e$ of $K_4$ by adding a pendant edge to a divalent vertex.

We say that $G$ is $H$-free if $G$ has no induced subgraph isomorphic to $H$. We write $\overline{G}$ to denote the complement of a graph $G$.

**Proof of Proposition 5.4.** From Fig. 2, it is easy to check that $W_4$, $W'_4$, a banner, a bull, a dart, an HVN and a kite are locally equivalent to $P_5$. Therefore, $G$ has no induced subgraph isomorphic to any of those graphs.

We may assume that $G$ is connected. If $G$ is $C_5$-free, then $G$ does not contain an odd hole because $G$ is $P_5$-free. Since $\overline{C_5}$ is isomorphic to $C_5$, $G$ is $\overline{C_5}$-free. In addition, since $\overline{W'_4}$ is the disjoint union of $P_2$ and $P_3$, $G$ is $\overline{C_k}$-free for every odd $k \geq 7$. Therefore $G$ is perfect by the strong perfect graph theorem [5].

Now we may assume that $G$ contains $C_5$ as an induced subgraph. Let $L_i$ be the set of vertices of $G$ having the distance $i$ to $C_5$. We may assume that $G$ is not $C_5$, that is, $L_1$ is not empty.

We claim that $L_1$ is complete to $L_0$. Suppose $v \in L_1$ is not complete to $L_0$. Then $v$ has exactly 1, 2, 3 or 4 neighbors in $L_0$. In each case it is easy to check that we can find an induced subgraph isomorphic to $P_5$, a bull, a banner or a kite, a contradiction.

Now we claim that $L_2 = \emptyset$. Suppose $v \in L_2$. Let $u \in L_1$ such that $uv$ is an edge. Now we see that $G$ contains a dart, a contradiction.

If two vertices $u, v$ in $L_1$ are adjacent, then $G$ contains a HVN as an induced subgraph, a contradiction. Thus, $L_1$ is stable.

If $L_1$ contains more than one vertex, then $G$ contains $W_4$, a contradiction. So $|L_1| = 1$, and so $G = W_5$. □
Finally let us discuss pivot-minors instead of vertex-minors. We may ask the following question, which is stronger than Question 1.4.

**Question 5.5.** Is it true that for every graph $H$, the class of graphs with no pivot-minor isomorphic to $H$ is polynomially $\chi$-bounded?

We will show that this is true if $H$ is a path. Here is a useful lemma, replacing Lemma 1.6.

**Lemma 5.6.** The graph $K_n \square \overline{K_n}$ has a pivot-minor isomorphic to $P_{n+1}$.

**Proof.** Let us remind that $\{a_1, a_2, \ldots, a_n\}$ is a clique and $\{b_1, b_2, \ldots, b_n\}$ is an independent set and $a_i$ is adjacent to $b_j$ if and only if $i \geq j$. It is easy to observe that $(K_n \square \overline{K_n}) \land a_2 b_2 \land a_3 b_3 \land a_4 b_4 \land \cdots \land a_{n-1} b_{n-1}$ has an induced path $a_1 a_2 a_3 \cdots a_n b_n$ of length $n$. □

So we deduce the following strengthening of Corollary 1.7 by the same proof.

**Proposition 5.7.** The class of graphs with no pivot-minor isomorphic to $P_n$ is polynomially $\chi$-bounded.

However, the analogue of Theorem 1.8 is false for pivot-minors. As mentioned in [2], if $k \not\equiv n \pmod{2}$, then $C_k$ has no pivot-minor isomorphic to $C_n$. This implies that there is a prime graph with an arbitrary long induced path having no pivot-minors isomorphic to $C_n$.

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**References**


