Electromagnetic-wave propagation through an array of superconducting qubits: Manifestations of nonequilibrium steady states of qubits

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We report a theoretical study of the propagation of electromagnetic waves (EWs) through an array of superconducting qubits, i.e., coherent two-level systems, embedded in a low-dissipation transmission line. We focus on the near-resonant case as the frequency of EWs $\omega \simeq \omega_q$, where ω_q is the qubit frequency. In this limit we derive the effective dynamic nonlinear wave equation allowing one to obtain the frequency-dependent transmission coefficient of EWs, $D(\omega)$. In the linear regime and a relatively wide frequency region we obtain a strong resonant suppression of $D(\omega)$ in both cases of a single qubit and chains composed of a large number of densely arranged qubits. However, in narrow frequency regions a chain of qubits allows the resonant transmission of EWs with greatly enhanced $D(\omega)$. In the nonlinear regime realized for a moderate power of applied microwave radiation, we predict and analyze various transitions between states characterized by high and low values of $D(\omega)$. These transitions are manifestations of nonequilibrium steady states of an array of qubits achieved in this regime.

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I. INTRODUCTION

The propagation of electromagnetic waves (EWs) in metamaterials—artificially prepared media composed of a network of interacting lumped electromagnetic circuits—has attracted recently enormous attention due to the variety of physical phenomena occurring in such systems, e.g., electromagnetically induced transparency (reflectivity) [1–3], "left-handed" metamaterials [4,5], and dynamically induced metastable states [6,7], just to name a few. These networks have been fabricated from metallic, semiconducting, magnetic, or superconducting materials.

The last case of networks based on superconducting elementary circuits presents a special interest because of an extremely low dissipation, a great tunabilty of the microwave resonances, and a strong nonlinearity [7–9]. In most of studied systems these superconducting electromagnetic circuits contained by one or a few Josephson junctions can be precisely described as classical nonlinear oscillators, and the interaction of propagating EWs with a network of such superconducting lumped circuits is determined by a set of classical nonlinear dynamic equations [7,10,11].

However, it has been well known for many years that small superconducting circuits can be properly biased in the *coherent macroscopic quantum regime*, and in a simplest case the dynamics of such circuits is equivalent to the quantum dynamics of two-level systems, i.e., qubits [12–18]. A surfeit of different types of superconducting qubits has been realized, e.g., dc-voltage-biased charge qubits [Fig. 1(a)] [12], flux qubits weakly [17] [Fig. 1(b)], and strongly [Fig. 1(c)] [2] interacting with a low-dissipation transmission line, transmons [13,18], etc.

As a next step these qubits are organized in different arrays or lattices forming *quantum electromagnetic networks*, and an inductive or capacitive coupling of such networks to an external low-dissipation transmission line allows one to experimentally access the frequency-dependent transmission coefficient of EWs, $D(\omega)$. The interaction of EWs with quantum networks of qubits results in a large number of coherent quantum phenomena on a macroscopic scale, e.g., collective quantum states [16–19], the observation of magnetically induced transparency [2], and coherent propagation of electromagnetic pulses [20,21], and the nonclassical states of photons [22] have been theoretically predicted and studied. Therefore, in this quickly developing field a natural question arises [19–23]: How does the coherent quantum dynamics of a network of superconducting qubits influence the EW propagation?

In this paper we present a systematic study of propagation of EWs through an array of qubits embedded in a low-dissipation transmission line (see Fig. 1). We will focus on the resonant case, i.e., $\omega \simeq \omega_q$, where ω_q is the qubit frequency, and the transmission coefficient $D(\omega)$ will be theoretically analyzed. In this paper we neglect completely the direct coupling between qubits and take into account the coupling between qubits and transmission line only. Such great simplification allows one to characterize the dissipation and decoherence of a whole quantum network with a single parameter, i.e., the relaxation time of a single qubit, T. With this assumption we reduce the low-dissipation dynamics of a qubits network to the dynamics of independent qubits exposed to the space- and time-dependent electromagnetic field. To obtain $D(\omega)$ we derive the effective nonlinear EW equation that is applicable in both limits, the low and high power of applied microwave radiation.

In the linear regime and a relatively wide frequency region near the resonance we obtain a strong suppression of $D(\omega)$ in

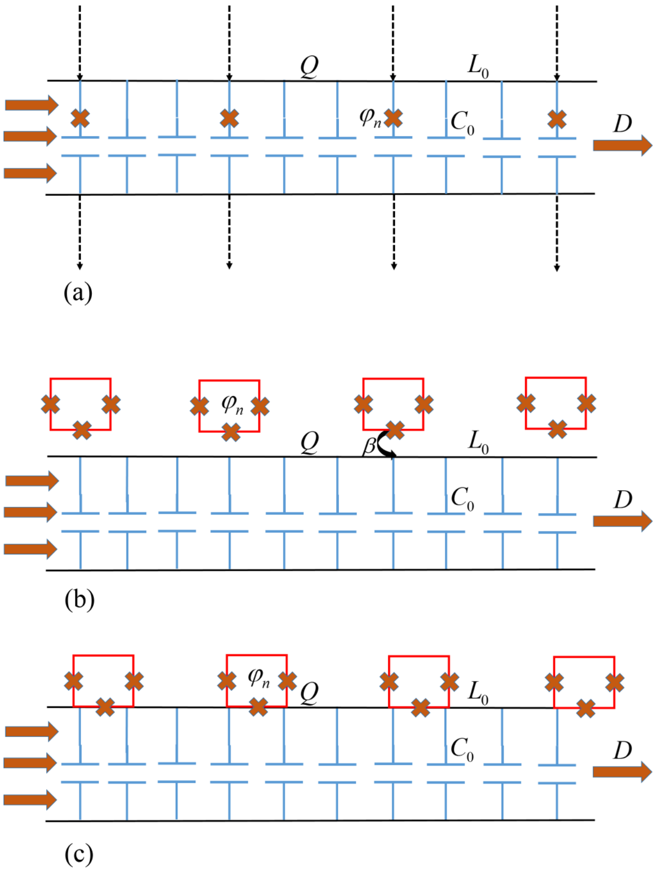

FIG. 1. Schematic of qubit arrays coupled to a low-dissipation transmission line: voltage-biased charge qubits (a), weakly coupled flux qubits (b), and strongly coupled flux qubits (c). Josephson junctions are illustrated by crosses; input and output of EWs are shown by arrows. The classical, Q, and quantum φ_n dynamic variables are shown. The properties of transmission line are characterized by two parameters: the capacitance C_0 and inductance L_0 per length. The D is the transmission coefficient of propagating EWs.

both cases of a single qubit and chains composed of a large number of densely arranged qubits. However, in a narrow frequency region for chains of qubits we obtain the resonant transmission of EWs with a greatly enhanced $D(\omega)$. As we turn to the nonlinear regime realized for a moderate power of applied microwave radiation, we predict and analyze various transitions between states characterized by high and low values of $D(\omega)$. We argue that these transitions are fingerprints of nonequilibrium steady states of an array of qubits.

The paper is organized as follows: In Sec. II we present our model for a qubits array embedded in a low-dissipation transmission line, introduce the Lagrangian, and derive the effective nonlinear wave equation for the electromagnetic field interacting with an array of qubits. In Sec. III we analyze the coherent quantum dynamics of a single qubit subject to an applied electromagnetic field in both limits of low and high power. In Sec. IV we apply the effective nonlinear wave equation derived in Sec. II to a study of the frequency-dependent transmission coefficient, $D(\omega)$, for a chain of densely arranged qubits. Moreover, we address both regimes, i.e., linear and nonlinear ones. Section V provides conclusions.

II. MODEL, LAGRANGIAN, AND DYNAMIC EQUATIONS

A. Model

Let us consider a regular one-dimensional array of N lumped superconducting quantum circuits embedded in a low-dissipation nondispersive transmission line (see Fig. 1). As the amplitude of propagating EWs is not too low, i.e., in the regime of a large number of photons, the electromagnetic field in the transmission line is characterized by coordinate- and time-dependent classical variables—the charge distribution, $Q(x,t)$. Different types of lumped superconducting quantum circuits have been realized (the schematics of arrays composed of charge [Fig. 1(a)] and flux [Figs. 1(b) and 1(c)] qubits are shown), and the quantum dynamics of such circuits is characterized by quantum variables—the Josephson phases, φ_n. An artificially prepared potential $U(\varphi_n)$ allows one to vary the circuits' resonant frequencies in a wide region. The dynamics of a whole system in the classical regime is described by a total Lagrangian, which consists of three parts: the Lagrangian of an electromagnetic field $L_{\rm EF}$, the Lagrangian of an array of lumped superconducting quantum circuits (qubits) $L_{\rm qb}$, and the interaction Lagrangian $L_{\rm int}$ describing the interaction between qubits and an electromagnetic field:

$$L = L_{\rm EW} + L_{\rm qb} + L_{\rm int}. \tag{1}$$

B. Lagrangian and dynamic equation

The electromagnetic field Lagrangian $L_{\rm EF}$ is written as

$$L_{\rm EF} = \frac{L_0 \ell}{2}\left\{\left[\frac{\partial Q}{\partial t}\right]^2 - c_0^2\left[\frac{\partial Q}{\partial x}\right]^2\right\}, \tag{2}$$

where $c_0 = 1/\sqrt{L_0 C_0}$ is the velocity of EWs propagating in the transmission line, and L_0 and C_0 are the inductance and capacitance of the transmission line per length, respectively; ℓ is the length of the system.

The Lagrangian of an array of lumped quantum circuits is written as

$$L_{\rm qb} = \sum_{n=1}^{N} \frac{E_J}{2\omega_p^2}(\dot{\varphi}_n - \dot{\phi}_{0n})^2 - U(\varphi_n), \tag{3}$$

where E_J, ω_p are the Josephson energy and the plasma frequency, respectively; the parameter $\dot{\phi}_{0n}$ is proportional to the gate voltage, and it allows one to vary the frequency of charge qubits [such gate circuits are shown by dashed arrows in Fig. 1(a)]. For charge qubits the potential $U(\varphi_n)$ is written explicitly as $U(\varphi_n) = E_J(1 - \cos\varphi_n)$, whereas for flux qubits [see Figs. 1(b) and 1(c)] the double-well potential reads as $U(\varphi_n) = -E_J[2\cos\varphi_n - \kappa\cos(2\varphi_n)]$, where κ is determined by an external magnetic flux and the critical currents of Josephson junctions.

The interaction part of the Lagrangian is derived by making use of a standard method [22,24,25], i.e., we start with the discrete model of the transmission line coupled to the array of qubits (see Fig. 1), write the dynamic equations (the Kirchhoff's current and voltage laws) for voltages V_n and currents I_n flowing in the nth cell of the transmission line taking into account the Josephson current flowing through the Josephson junctions of qubits, transform the set of difference equations to the differential dynamic equation, and finally construct the interaction Lagrangian, L_{int}. Such derivation has been done explicitly in numerous works for different types of qubits—see, e.g., Refs. [22,24–27]—and the interaction Lagrangian is written as

$$L_{\text{int}} = -\frac{\hbar\alpha w}{2e}\sum_{n=1}^{N}\delta(na-x)Q(t,x)\dot{\varphi}_n, \tag{4}$$

where the coupling coefficient α varies by around five orders of magnitude from 10^{-2} for a weak coupling between qubits and transmission line [17] [Fig. 1(b)] up to 4×10^2 for a so-called ultrastrong coupling regime [2,28,29] [Fig. 1(c)]. Such an ultrastrong coupling regime has been achieved in systems where the superconducting qubits are directly incorporated in the transmission line, and in this case the coupling coefficient is determined by the ratio of the Josephson inductance to the geometrical inductance. The charge qubits [Fig. 1(a)] and flux qubits [Figs. 1(b) and 1(c)] are capacitively and inductively coupled to the transmission line accordingly. Here w is the geometrical size of lumped quantum circuits (qubits), $w \ll \ell$.

As we turn to the coherent quantum regime of networks of qubits, the dynamics of EWs is described by the specific wave equation as

$$\frac{\partial^2 Q}{\partial t^2} - \gamma\frac{\partial Q}{\partial t} - c_0^2\frac{\partial^2 Q}{\partial x^2} = -\frac{\hbar}{2e}\frac{\alpha w}{L_0\ell}\sum_{n=1}^{N}\delta(na-x)\langle\dot{\varphi}_n\rangle_{\text{eq}}, \tag{5}$$

where we take into account the dissipation effects in the transmission line characterized by the phenomenological parameter $\gamma \ll 1$. Here $\langle\cdots\rangle_{\text{eq}}$ denotes the quantum mechanical averaging over the equilibrium state of the quantum network.

C. Quantum dynamics of a single qubit and effective wave equation

Since we neglect the direct coupling between elementary circuits, the coherent quantum dynamics of a network is reduced to the sum of an independent lumped electromagnetic circuit exposed to an applied electromagnetic field. The quantum dynamics of a single element is determined by the time-dependent Hamiltonian, $\hat{H}_{\text{qb}} = \hat{H}_0 + \hat{H}_t$, where the equilibrium Hamiltonian $\hat{H}_0$ is

$$\hat{H}_0 = \frac{\omega_p^2}{2E_J}(\hat{p}_\varphi - p_0)^2 + U(\varphi), \tag{6}$$

and the nonequilibrium part of the total Hamiltonian, explicitly depending on time, $\hat{H}_t$ is

$$\hat{H}_t = \frac{\hbar\alpha\omega_p^2}{2eE_J}Q(t,x)\hat{p}_\varphi. \tag{7}$$

In the resonant regime as the EW frequency $\omega \simeq \omega_q$, we truncate the explicit Hamiltonian [see Eqs. (6) and (7)] to the Hamiltonian of two-level systems. In all cases (see schematics in Fig. 1) these two levels can be fine-tuned to the resonance with the frequency of EWs propagating in the transmission line, and the effective Hamiltonian is written as

$$\hat{H}_{\text{eff}} = \frac{\omega_q}{2}\hat{\sigma}_z + \frac{(\hbar\omega_p)^2\alpha}{4eE_J}Q(x,t)\hat{\sigma}_x, \tag{8}$$

where $\hat{\sigma}_{x,z}$ are the Pauli matrices. Here the qubit frequency ω_q is determined explicitly by physical parameters of corresponding elementary circuits. In particular, for the charge qubits case shown in Fig. 1(a), at the avoid-crossing point the qubit frequency ω_q is expressed as $\omega_q = E_J$ [12].

The time-dependent wave function of a charge qubit is written as

$$\Psi(t) = C_-(t)f_- + C_+(t)f_+, \tag{9}$$

where $f_\pm = \frac{1}{\sqrt{4\pi}}(1 \pm e^{i\varphi})$ are stationary wave functions of two states. The corresponding quantum-mechanical average of the operator $\langle\dot{\varphi}\rangle$ on the right-hand side of Eq. (5) reads as

$$\langle\dot{\varphi}\rangle_{\text{eq}} = \frac{\hbar\omega_p^2}{E_J}\Re e[C_-(t)C_+^*(t)]. \tag{10}$$

Taking into account the initial conditions $C_-(0) = 1$ and $C_+(0) = 0$, and using the resonant condition, $\omega_q \simeq \omega$, we obtain in the nondissipation (nd) regime

$$S_n^{\text{nd}}(\omega) = \int dt e^{i\omega t}\Re e[C_-(t)C_+^*(t)]$$
$$= \eta q(x_n,\omega)\frac{1-\omega/\omega_q}{(1-\omega/\omega_q)^2 + \eta^2|q(x_n,\omega)|^2},$$

where we introduce the dimensionless strength of interaction, $\eta = \alpha[\hbar\omega_p/(2E_J)]^2$, and the dimensionless charge distribution, $q(x,t) = Q(x,t)/e$.

In the low-dissipation regime the Fourier transforms of time-dependent quantum-mechanical averages are expressed through the corresponding Fourier transforms of the time-dependent quantum mechanical correlation function of the nth qubit, $S_n(\omega) = \int dt e^{i\omega t}\langle\varphi_n(t)\varphi_n(0)\rangle$ and the charge distribution $q(x_n,\omega)$ [30,31] as

$$\langle\dot{\varphi}\rangle_{\text{eq}} = \frac{\hbar\omega_p^2}{E_J}S_n(\omega)q(x_n,\omega). \tag{11}$$

Introducing the relaxation time T *of a single qubit*, and by solving the dynamic equations for the density matrix (the Maxwell-Bloch equation) with the help of the rotation wave approximation (RWA) [32], we obtain the Fourier transform of the correlation function of the nth qubit as

$$S_n(\omega) = \eta\frac{1-\omega/\omega_q + i/(\omega_q T)}{(1-\omega/\omega_q)^2 + 1/(\omega_q T)^2 + \eta^2|q(x_n,\omega)|^2}. \tag{12}$$

Substituting (11) and (12) in (5) we obtain the effective equation allowing one to analyze the transmission coefficient $D(\omega)$ for propagating EWs of frequency ω:

$$c_0^2\frac{d^2q}{dx^2} + \omega^2 q(x) + i\gamma\omega_q(x)$$
$$= \frac{2w\hbar\omega_q}{e^2L_0\ell}\eta^2\sum_{n=1}^{N}\delta(na-x)$$
$$\times\frac{1-\omega/\omega_q + i/(T\omega_q)}{\eta^2|q(x,\omega)|^2 + (1-\omega/\omega_q)^2 + 1/(T\omega_q)^2}q(x,\omega). \tag{13}$$

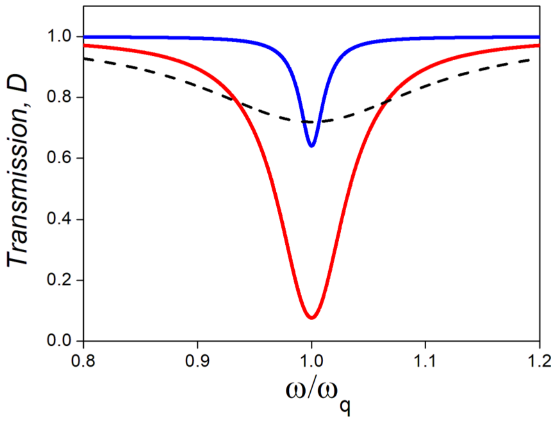

FIG. 2. The transmission of EWs, $D(\omega)$: the linear regime, a single qubit embedded in the transmission line. The parameters were chosen as $\Gamma = 10^{-2}$, $g = 0.06$ [red (light gray) solid line], $\Gamma = 10^{-2}$, $g = 0.008$ [blue (gray) solid line], $\Gamma = 10^{-1}$, $g = 0.06$ (dashed line).

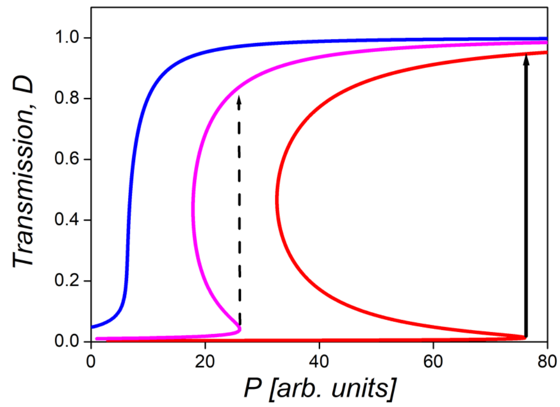

FIG. 3. The transmission coefficient D on the power P_0 of applied microwave radiation. The parameters were chosen as $\omega = \omega_q$ and $g/\Gamma = 9$ [blue (dark gray) line], $g/\Gamma = 16$ [magenta (gray) line], $g/\Gamma = 34.6$ [red (light gray) line].

III. EW TRANSMISSION: A SINGLE QUBIT

In this section we consider the EW transmission through a single qubit. The charge distribution $q(x, \omega)$ satisfies the effective equation

$$c_0^2\frac{d^2q}{dx^2} + [\omega^2 + i\gamma\omega]q(x) = \frac{2\hbar w\omega_q}{e^2L_0\ell}\eta^2\delta(x)\frac{1-\omega/\omega_q + i/(T\omega_q)}{\eta^2|q(x,\omega)|^2 + (1-\omega/\omega_q)^2 + 1/(T\omega_q)^2} \times q(x,\omega). \tag{14}$$

As the power of EWs is small, i.e., $|Q(x)/e| \ll (\eta\omega_q T)^{-1}$, the transmission coefficient $D(\omega)$ reads as

$$D(\omega) = \left[1 + \frac{g}{4}\frac{g+4\Gamma}{(\omega/\omega_q - 1)^2 + \Gamma^2}\right]^{-1}. \tag{15}$$

Here we introduce the dimensionless relaxation rate of a single qubit, $\Gamma = (T\omega_q)^{-1}$, and the interaction strength, $g = 2\eta^2\frac{\hbar w}{e^2 c_0 L_0\ell}$, where w is the geometrical size of the lumped quantum circuit (qubit), $w \ll \ell$. The dependencies of $D(\omega)$ in the linear regime for different values of Γ and g are presented in Fig. 2. A most important effect is a strong resonant suppression of EW propagation in the limit of $g/\Gamma \gg 1$. The width of the $D(\omega)$ curve is diminished as the relaxation rate Γ decreases. Such strong suppression of the transmission probability $D(\omega)$ for the EWs interacting with a single qubit has been observed in numerous experiments; see, e.g., Refs. [33,34]. As we turn to the high-power regime of applied microwave radiation, i.e., $|q(-\infty)| \simeq \sqrt{P_0} \gg \Gamma/\eta$ we obtain the transmission coefficient as a solution of the transcendent equation

$$D(\omega) = \left\{1+\frac{g}{4}\frac{(4\Gamma+g)[(\omega/\omega_q-1)^2+\Gamma^2]+4\Gamma\eta^2P_0D(\omega)}{[(\omega/\omega_q-1)^2+\eta^2P_0D(\omega)+\Gamma^2]^2}\right\}^{-1}. \tag{16}$$

Here P_0 is the power of applied microwave radiation far from the resonator. An analysis of Eq. (16) shows that in the nonlinear regime the transmission coefficient D is determined strongly by the ratio of two parameters g and $\sqrt{(\omega-\omega_q)^2+\Gamma^2}$. Indeed, if $g/\sqrt{(\omega-\omega_q)^2+\Gamma^2} \leqslant 1$ the transmission coefficient just monotonically increases with P_0, but in the opposite case, $g/\sqrt{(\omega-\omega_q)^2+\Gamma^2} \gg 1$, there is a particular range of power P_0, where two dynamic states of EWs characterized by large and small transmission coefficients are obtained. The numerically calculated dependencies of the transmission coefficient on the power P_0 are shown in Fig. 3. Obtained in this paper, the dynamic bistability has a physical origin in the ac-field-induced saturation effect in the quantum-mechanical dynamics of two-level systems (qubits) and strongly resembles the classical nonlinear bistability observed previously in single Josephson junctions [35] or SQUIDs [6,7].

IV. EW TRANSMISSION: A PERIODIC ARRAY OF SUPERCONDUCTING QUBITS

In this section we consider the propagation of EWs through a periodic array of N qubits. In this case the charge distribution $q(x, t)$ is determined by Eq. (13).

By making use of the method elaborated for the solution of the Schrödinger equation with the Kronig-Penney potential [36] we present the charge distribution $q(x)$ in the following form:

$$q(x) = -\frac{i}{2kc_0^2}\sum_n \beta\{q_n\}q_n \exp(ik|x-x_n|),$$
$$\beta = \eta^2\frac{2\hbar}{e^2}\frac{w\omega_q}{L_0\ell}\frac{1-\omega/\omega_q+i/(T\omega_q)}{(1-\omega/\omega_q)^2+1/(T\omega_q)^2+\eta^2|q(x,\omega)|^2}, \tag{17}$$

where the wave vector $k = \sqrt{\omega^2+i\gamma\omega}/c_0$, and $q_n = q(x_n)$ is the amplitude of the propagating charge distribution at the point of x_n. By making use of the properties of the δ function, we obtain the set of discrete equations for q_n:

$$q_{n+1} + q_{n-1} - \left[2\cos ka - \frac{\beta\{q_n\}}{kc_0^2}\sin(ka)\right]q_n = 0\,. \qquad (18)$$

Here $a = \ell/N$ is the distance between the adjacent qubits in the array.

Next, we study the EW propagation through an array of densely arranged qubits, i.e., as the condition $ka \ll 1$ is valid. In this case one can transform the difference equation (18) into the differential equation

$$c_o^2\frac{d^2q(x)}{dx^2} + \left[\omega^2 + i\gamma\omega + \frac{\beta\{q(x)\}}{a}\right]q(x) = 0. \qquad (19)$$

The transmission coefficient is determined as $D(\omega) = |q(\ell)/q(0)|^2$.

A. Low-power regime

As the power of applied microwave radiation is low, one can neglect the nonlinear dependence of $\beta(q)$ on q, and by making use of a well-known result for the quantum tunneling through a rectangular barrier the transmission coefficient is written as

$$D = \left|\cos[k\ell\sqrt{K(\omega)}] + \frac{i}{2}\sqrt{K(\omega)}\sin[k\ell\sqrt{K(\omega)}]\right|^{-2}, \qquad (20)$$

where $K(\omega) = \frac{gc_0}{\omega_q a}\frac{(1-\omega/\omega_q)+i\Gamma}{(1-\omega/\omega_q)^2+\Gamma^2}$. For a densely arranged array of qubits the transmission coefficient $D(\omega)$ is determined by a single parameter g/a, and, therefore, there is a particular equivalency between two setups: a moderate number of qubits strongly coupled to the transmission line and a large number of qubits moderately coupled to the transmission line.

Next, we consider a regime most relevant to current experiments as the total length of a system is smaller than the wave length of EWs, i.e., $\ell \ll \lambda = c_0/\omega$. For typical values of transmission line parameters, i.e., $\ell = 100\ \mu$m, $\omega_q = 2\pi \times 5$ GHz, one can obtain $kl \simeq 0.01$ and embed in the transmission line up to $N = 20$ flux qubits (or even more charge qubits). We assume also that the effective strength of interaction between a single qubit and EWs is large, i.e., $\beta/a \gg \omega_q^2$. With such assumptions the dependencies of $D(\omega)$ for different values of an effective strength of interaction g are presented in Fig. 4. Beyond a standard resonant suppression of $D(\omega)$ observed for moderately large values of g [see Fig. 4, red (light gray) line], we obtain a great enhancement of $D(\omega)$ in an extremely narrow region of frequencies [see Fig. 4, blue (dark gray) line]. This effect of resonant transparency of EWs propagating through an array of qubits occurs for an extremely large values of an effective strength of interaction g. We notice that such resonant propagation of EWs through a chain of densely arranged superconducting qubits has been experimentally observed in Ref. [2] where the experiments were performed on superconducting arrays composed of $N = 15$ qubits, and large values of coupling were achieved by direct incorporation of the Josephson junctions of qubits in the superconducting transmission line. Moreover, as the size of the array increases we obtain a large set of peaks in the dependence of $D(\omega)$. It is shown in Fig. 5.

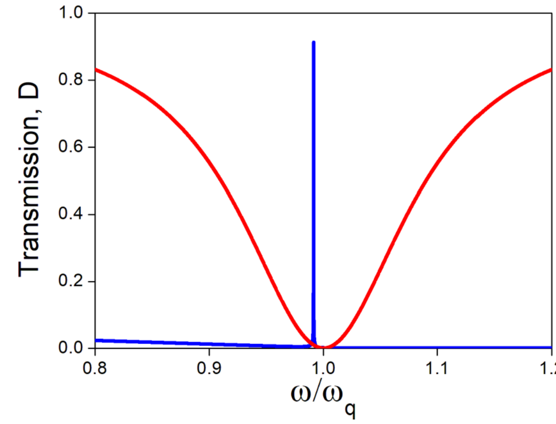

FIG. 4. The transmission coefficient of EWs, $D(\omega)$: the linear regime, a moderate size ($k\ell = 0.01$) array of qubits embedded in a low-dissipation transmission line. The parameters were chosen as $\Gamma = 3 \times 10^{-3}$ and $gc_0/(\omega_q a) = 9$ [red (light gray) solid line], $gc_0/(\omega_q a) = 900$ [blue (dark gray) solid line].

B. High-power regime

In the regime of high-power applied microwave radiation the dynamics of EWs is determined by the generic Eq. (19) written as

$$\frac{d^2q}{dx^2} + \left[k^2 + \frac{\chi}{|q|^2 + \xi^2}\right]q(x) = 0, \qquad (21)$$

where $\chi = [gc_0/(\eta^2\omega_q a)](\omega_q - \omega)$ and $\xi = (1/\eta^2)[(\omega_q - \omega)^2 + \Gamma^2]$. Here we neglect a small absorption of EWs, i.e., an imaginary part of k and χ. We solve such an intrinsically nonlinear wave equation by making use of an analogy with the famous Kepler problem in classical mechanics [37]. To do that we introduce the spatially dependent amplitude $r(x)$ and the phase $\phi(x)$ of EWs as $q(x) = r(x)e^{i\phi(x)}$ and $|q| = r$. Spatial distributions of the amplitude $r(x)$ and phase $\phi(x)$ of the electromagnetic field

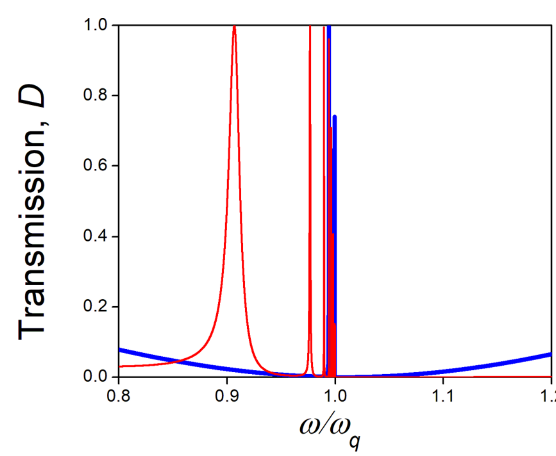

FIG. 5. The transmission coefficient of EWs, $D(\omega)$: the linear regime, large size arrays of qubits embedded in a low-dissipation transmission line. The parameters were chosen as $\Gamma = 3 \times 10^{-3}$, $gc_0/(\omega_q a) = 9$, and different values of $k\ell = 0.08$ [blue (dark gray) thick line], $k\ell = 0.32$ [red (light gray) thin line].

are determined by following equations:

$$r^2\frac{d\phi}{dx} = C, \quad \frac{d}{dx}\left[(r')^2 + \frac{C^2}{r^2}\right] + (r^2)'[k^2 + R(r)] = 0, \tag{22}$$

where we introduce the nonlinear function $R(r) = \frac{\chi}{r^2+\xi^2}$, and C is the constant that has to be found from the boundary conditions. The boundary conditions are derived from the continuity of the electric and magnetic fields of EWs at the boundaries, $x = 0$ and $x = \ell$, of a system. The boundary conditions are explicitly written as

$$\frac{d}{dx}\ln q(\ell) = ik, \quad A + B = q(0), \quad A - B = q'(0)/ik, \tag{23}$$

where the amplitude of incident EWs, $A \propto \sqrt{P}$ and P, is the power of an incident EW. The transmission coefficient of propagating EWs is determined as $D = |q(\ell)/A|^2$. The solution of Eq. (22) is obtained as

$$\int_{r(0)}^{r(\ell)} \frac{du}{\sqrt{E - \frac{C^2}{r^2} - r^2k^2 - \chi\ln(r^2+\xi^2)}} = \ell, \tag{24}$$

where the constant E is the effective energy of a system. The constant C is determined as: $C = k[r(\ell)]^2$. In the limit of not extremely large coupling g and large system size, $kl \gg 1$, using the condition $|r(\ell) - r(0)| \ll r(0)$ we write the expression for the transmission coefficient $D(\omega)$ as

$$D^{-1} = 1 + \frac{\chi}{2k^2}\frac{1-z}{\xi^2 + r^2(\ell)}, \tag{25}$$

where the variable $z = r(0)/r(\ell)$ is close to one. The parameter z is determined by the transcendent equation derived from Eq. (24) as

$$\sqrt{\frac{[r^2(\ell)+\xi^2]}{2\chi}}\int_z^1 \frac{dy}{\sqrt{1-y}} = \ell. \tag{26}$$

Thus, the parameter z is obtained explicitly as

$$1 - z = \frac{\chi\ell^2}{2[r^2(\ell)+\xi^2]}. \tag{27}$$

Substituting (27) in (25) we obtain in a strongly nonlinear regime the transmission coefficient of $D(\omega)$ as

$$\frac{1}{D} = 1 + \left[\frac{\chi\ell}{2k\xi^2(PD+1)}\right]^2. \tag{28}$$

For various parameters $\omega_q - \omega$, χ, and Γ the dependencies of $D(P)$ are presented in Fig. 6. Thus, the main result of this section is that if for low-power EWs the transmission D is strongly suppressed ($D \ll 1$), in the high-power limit the transmission will be recovered to $D \simeq 1$. The origin of such effect is an equalizing of the populations of qubit states in the limit of a large power of EWs, that, in turn, strongly suppresses the ac response of qubits to the applied electromagnetic field.

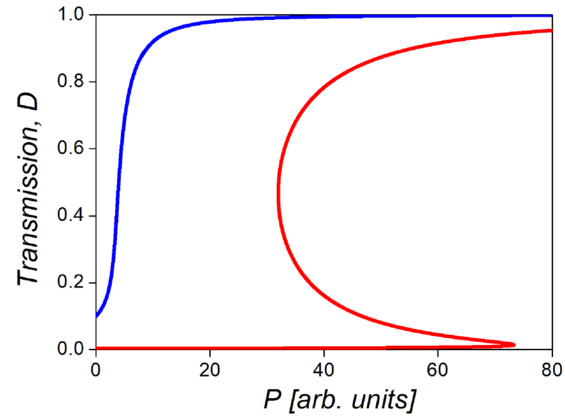

FIG. 6. The transmission of EWs through an array of qubits: high-power regime. The different values of parameter $\chi\ell/(4k\xi^2)$ are chosen as 4 [blue (dark gray) line] and 17 [red (light gray) line].

V. CONCLUSION

In conclusion we theoretically studied the propagation of EWs through a one-dimensional array of densely arranged superconducting qubits, i.e., coherent two-level systems embedded in a low-dissipation transmission line (see Fig. 1). A particular near-resonant case as $\omega \simeq \omega_q$ has been studied. We derive an effective nonlinear wave equation taking into account a nonequilibrium state of qubits, Eq. (13).

The dependencies of transmission coefficient $D(\omega, P)$ on the frequency ω and power P of applied microwave radiation were obtained. In particular, for both cases of a single qubit and large arrays of qubits the resonant suppression of $D(\omega)$ was found in the limit of small power P and as $|\omega - \omega_q| \ll \omega_q$ (see Figs. 2 and 4). However, the resonant transmission with $D \simeq 1$ was found in large arrays of qubits for an extremely large coupling of qubits with EWs (see Figs. 4 and 5) and in a narrow band of frequencies. Notice here that the effect of resonant transmission of EWs through an array of qubits has been observed in Ref. [2].

In the limit of high powers of applied EWs the large transmission $D \simeq 1$ was recovered in both cases of a single qubit and an array of qubits.

We anticipate that strong variations of transmission coefficient D on the frequency and power of EWs will be used in electronic devices with quantum efficiency.

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