Bound states of little strings and symmetric orbifold conformal field theories

Ambreen Ahmed,1 Stefan Hohenegger,2,3 Amer Iqbal,1,3,4 and Soo-Jong Rey3,5

1Abdus Salam School of Mathematical Sciences, G.C. University, Lahore 54600, Pakistan
2Université de Lyon UMR 5822, CNRS/IN2P3, Institut de Physique Nucléaire de Lyon, 4 rue Enrico Fermi, 69622 Villeurbanne Cedex, France
3Fields, Gravity & Strings, CTPU, Institute for Basic Sciences, Daejeon 34047, Korea
4Center for Theoretical Physics, Lahore 54600, Pakistan
5School of Physics and Astronomy & Center for Theoretical Physics, Seoul National University, Seoul 08826, Korea

(Received 28 June 2017; published 9 October 2017)

We study BPS bound states of little strings in a limit where they realize monopole strings in five dimensional gauge theories. The latter have gauge group $U(M)^N$ and arise from compactification of $(1,0)$ little string theories of type $A_{N-1} \times A_{N-1}$. We find evidence that the partition function of a certain subclass of monopole strings of charge $(k,\ldots,k)$ ($k \geq 1$) is expressible as the partition function of a symmetric orbifold sigma model, whose target space is precisely the symmetric product of the moduli space of monopoles with charge $(1,\ldots,1)$.

DOI: 10.1103/PhysRevD.96.081901

Little string theories (LSTs) refer to a quantum theory of noncritical strings without gravity, living in six spacetime dimensions. It has been noted that their properties are similar to those of strings encountered in other physical applications, such as hadronic strings in QCD and magnetic flux tubes in superconductors. It has also been noted that little string theories provide microstate descriptions of various supersymmetric black holes in four dimensions. When compactified to five dimensions, the theory gives rise to monopole strings [1–3], whose various properties can be studied using the underlying little strings [4]. Monopole strings, as the name indicates, are stringlike solutions in five dimensions which appear pointlike and carry monopole charge in the transverse three dimensions [5,6]. All these connections motivate a deeper study of the structure of LSTs.

LST of type $A_{N-1} \times A_{N-1}$ with $N = (1,0)$ supersymmetry can be engineered using $N$ M5-branes probing a transverse $A_{N-1}$ orbifold geometry. The M-theory background is given by $\mathbb{R}^4 \times \mathbb{T}^2 \times S^1 \times \mathbb{R}^4/Z_M$, where the M5-branes are extended along $\mathbb{R}^4 \times \mathbb{T}^2$ and separated along $S^1$. In a series of papers [4,7–9], such LSTs of type $A_{N-1}$ probing a $Z_M$ orbifold background were studied and their partition functions were calculated using a dual setup of D5- and NS5-branes in type IIB string theory. The latter in turn is dual to a particular class of toric Calabi-Yau threefolds $X_{N,M} \sim X_{1,1}/Z_M \times Z_N$ where $X_{1,1}$ resembles the resolved conifold near certain boundaries of the moduli space. The partition function of the little strings is refined by the insertion of three $U(1)$ currents: $U(1)_\varepsilon$ corresponding to a rotation of the transverse $\mathbb{R}^4$ [8] and $U(1)_{\epsilon_{1,2}}$ acting on $\mathbb{R}^4 \sim \mathbb{C}^2$ as $(z_1,z_2) \mapsto (e^{i\varepsilon_1}z_1,e^{i\varepsilon_2}z_2)$. When one of the world-volume directions is compactified, the corresponding five-dimensional world-volume theory on the M5-branes becomes a gauge theory with gauge group $U(M)^N$ broken down to $U(1)^{NM}$. The five-dimensional theory on $\mathbb{R}^3 \times \mathbb{T}^2$ contains monopole strings coming from M2-branes stretched between the M5-branes [2,3] and wrapped on $\mathbb{T}^2$. Thus, the partition function of little strings calculates the $\mathcal{N} = (2,0)$ elliptic genus of monopole strings, in the limit $\varepsilon_2 \mapsto 0$ which is required by the compactification of the corresponding direction on which $U(1)_{\varepsilon_2}$ acts. These monopole strings carry charges $k_i|i = 1,\ldots,N$ (which is the number of M2-branes between the $i$-th and $i+1$-th M5-brane) as well as fractional momenta $p_a|a = 1,\ldots,M$ along the transverse $S^1$.

In this paper, we study a subsector of the BPS spectrum consisting of states such that $k_1 = \cdots = k_N = k$ and $p_1 = \cdots = p_M = p$. Namely, this subsector with charges $(k,\ldots,k)$ contains $k$ M2-branes starting and ending on any given M5-brane. We show that the degeneracies of these BPS states with quantum numbers $(k,p)$ only depend on the product $kp$ and therefore degeneracies of higher charge monopole strings of this type are completely determined by degeneracies of those with charge one. We further argue that the partition function of these monopole strings can be expressed in terms of a conformal field theory (CFT) whose target space is a symmetric orbifold.

The partition function of this class of LSTs, which we denote by $Z_{N,M}$, is given by [3,9,10]

$$Z_{N,M} = (W(T,m,e_\pm))^NZ_{N,M}(T,T,m,e_\pm),$$

where $T = (t_1,\ldots,t_N)$ with $t_i$ (with $i = 1,\ldots,N$) being the separation between the $i$-th and the $(i+1)$-th M5-brane and $T = (T_1,\ldots,T_M)$ determine the charge of the states with respect to the $Z_M$ orbifold. Furthermore, $m$ and $e_\pm = e^{i\frac{\pi}{2}}$ correspond to the $U(1)$s we discussed above, which are required to render $Z_{N,M}$ well-defined.
The partition function (1) receives contributions from two parts. The first part \( W(T, m, \epsilon_\pm) \) is the contribution of a single M5-brane wrapped on a circle with transverse \( \mathbb{Z}_M \) orbifold. This contribution can be determined by studying the modes of a single six-dimensional abelian tensor multiplet on a circle colored by the transverse orbifold. The second factor \( \mathbb{Z}_{N,M} \) is the contribution coming from M2-branes suspended between the M5-branes. Since these M2-branes are the little strings in six dimensions, this second factor contains the little string and monopole string degeneracies and will be the object of study in this paper. It can be expanded in the following fashion

\[
\mathcal{Z}_{N,M}(t, T, m, \epsilon_\pm) = \sum_{k_1, \cdots, k_N} \mathcal{Q}_1^{k_1} \cdots \mathcal{Q}_N^{k_N} \mathcal{Z}_{N,M}^{k_1 \cdots k_N},
\]

where \( \mathcal{Q}_i = e^{-i \theta} \) and \( \mathcal{Z}_{N,M}^{k_1 \cdots k_N}(T, m, \epsilon_{1,2}) \) captures the BPS states of \( k_i \) M2-branes suspended between the \( i \)-th and the \( (i+1) \)-th M5-brane, respectively.

The free BPS energy associated with the partition function (1) is given by \( F_{N,M} = \ln(\mathcal{Z}_{N,M}) \). In this paper, however, we study \( (\mathcal{Q}_a = e^{-T_a}) \)

\[
\mathcal{F}_{N,M} = \sum_{k_1, \cdots, k_N} \mathcal{Q}_1^{k_1} \cdots \mathcal{Q}_N^{k_N} \int \frac{dQ_1d\mathcal{Q}_1}{Q_1^{k_1+1}Q_1^{k_1+1}} \ln \mathcal{Z}_{N,M},
\]

which only counts contributions from the aforementioned subspace of the BPS Hilbert space in which every M5-brane has the same number of M2-branes starting and ending on it. This reduced free energy \( \mathcal{F}_{N,M} \) can be written as the infinite sum

\[
\mathcal{F}_{N,M}(\rho, \tau, m, \epsilon_{1,2}) = \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{G}_{N,M}(n \rho, n \tau, nm, n \epsilon_\pm)
\]

and only depends on \( (\tau, \rho, m, \epsilon_\pm) \), which are sums of all \( T \) and \( t \), respectively:

\[
\tau = \frac{i}{2\pi} (T_1 + \cdots + T_M) \quad \text{and} \quad \rho = \frac{i}{2\pi} (t_1 + \cdots + t_N).
\]

Furthermore, \( \mathcal{G}_{N,M}(\rho, \tau, m, \epsilon_\pm) = \sum_{k_1, \cdots, k_N} \mathcal{Q}_1^{k_1} \mathcal{G}_M^{(k_1, \cdots, k_N)} \) where the superscript contains \( N \) entries of \( k \) captures the BPS degeneracies of \( k \) M2-branes stretched between any adjacent pair of the \( N \) M5-branes and winding \( k \) times around the transverse \( S^1 \). These M2-branes carry arbitrary momenta along the \( S^1 \) in M5-brane world volume.

For \( N = M = 1 \), the free energy \( \mathcal{F}_{1,1} \) captures all the BPS states in the theory

\[
\mathcal{F}_{1,1}(\tau, \rho, m, \epsilon_\pm) = \sum_{k > 0} \mathcal{Q}_1^{k} \mathcal{G}_1^{(k)}(\tau, \rho, m, \epsilon_\pm),
\]

where \( \mathcal{G}_1^{(k)} \) can be expressed in terms of the contribution of a single M2-brane winding once around \( S^1 \)

\[
G_1^{(k)} = \frac{1}{k} \sum_{a=0}^{k-1} G_1^{(1)} \left( \frac{\tau + a}{k}, m, \epsilon_\pm \right).
\]

Here, \( G_1^{(1)} \) can be written as the following quotient of Jacobi theta-functions

\[
G_1^{(1)}(\tau, m, \epsilon_\pm) = \frac{\theta_1(\tau, m + \epsilon_-)}{\theta_1(\tau, \epsilon_+ + \epsilon_-)} \frac{\theta_1(\tau, m - \epsilon_-)}{\theta_1(\tau, \epsilon_+ - \epsilon_-)}.
\]

We see from the Fourier expansion of \( G_1^{(k)} \)

\[
G_1^{(k)} = \sum_{n, c, r, s} c_k(n, \epsilon', r, s) e^{2\pi i n q' t^s} (q')^s
\]

where \( q = e^{i\epsilon_1} \) and \( t = e^{-i\epsilon_2} \) that relation (4) implies

\[
c_k(n, \epsilon', r, s) = c_1(kn, \epsilon', r, s).
\]

In fact, the relation (4) reflects that \( \mathcal{Z}_{1,1} \) is the partition function of a two-dimensional \( N = (2, 2) \) supersymmetric sigma model whose target space is a symmetric product; i.e., it is the partition function of a symmetric orbifold theory and therefore it can be expressed as [11]

\[
\mathcal{Z}_{1,1} = \prod_{k,n,r,s} (1 - \mathcal{Q}_k^{A_+} \mathcal{Q}_k^{A_-} \mathcal{Q}_k^{B_+} \mathcal{Q}_k^{B_-} c_1(kn, \epsilon', r, s)) = \sum_{k=0}^{\infty} \mathcal{Q}_k^{A_+} \mathcal{Q}_k^{B_+} \mathcal{Z}((C^2)^k/S_k),
\]

where \( \mathcal{Z}((C^2)^k/S_k) \) is the (equivariantly regularized) elliptic genus of \( (C^2)^k/S_k \) which can be written as

\[
\mathcal{Z}((C^2)^k/S_k) = \frac{1}{k!} \sum_{h \equiv gh/h \in S_k} g^n \mathcal{H}_k(G_1^{(1)}).
\]

In the sum, \( g^n \) denotes the \( N' = (2, 2) \) partition function of a sigma model whose target space is the product of \( k \) copies of \( C^2 \) and whose world-sheet boundary conditions in the space and time directions are twisted by \( h \) and \( g \), respectively. It follows from Eqs. (3) and (4) that

\[
\ln(\mathcal{Z}_{1,1}) = \sum_{k=1}^{\infty} \mathcal{Q}_k^{A_+} \mathcal{H}_k(G_1^{(1)}).
\]
The function \( \mathcal{H}_k(f_wz(\tau, \bar{\tau})) \) in turn is a Jacobi form of index \( rk \) and weight \( w \). Notice that, under \( SL(2, \mathbb{Z}) \) transformations, \( G_1^{(1)} \) transforms as

\[
G_1^{(1)} \left( \frac{1}{2} \frac{m}{\tau} + e_\pm \right) = e^{\mp \frac{\pi i}{6}(m^2 - e_\pm)} G_1^{(1)}(\tau, m, e_\pm);
\]

i.e., it has weight \( w = 0 \) and index \( k = (1, -1, 0) \) with respect to \((m, e_+, e_-)\).

With Eqs. (8), (9) and (10), we have three representations of \( Z_{1,1} \) all of which follow from each other and are a consequence of the relation (4). In the following, we shall find a generalization of the latter for the particular BPS subsector contributing to \( \mathcal{F}_{N,M} \) (2) in more complicated \( M \)-brane configurations: indeed, for arbitrary \((N,M)\), a specific part of the partition function can be identified with the elliptic genus of a product of instanton moduli spaces and can again be written as the partition function of a symmetric orbifold theory. The free energy \( \mathcal{F}_{N,M} \) (2) is given in terms of \( G_{M}^{(k,...,k)} \) by

\[
\mathcal{F}_{N,M} = \sum_{n \in \mathbb{N}} \frac{1}{n} \sum_{k_1 \geq 1} Q^k_{n} G_{M}^{(k,...,k)}(n\tau, nm, n\epsilon_\pm) = \frac{1}{k} \sum_{n \in \mathbb{N}} Q^k_{n} G_{M}^{(k,...,k)}(n\tau, nm, n\epsilon_\pm).
\]  

The functions \( G_{M}^{(k,...,k)}(\tau, m, e_\pm) \) diverge in the limit \( \epsilon_{1,2} \rightarrow 0 \) proportional to \( \frac{1}{i\epsilon_{1,2}} \), and so we can define the slightly modified Nekrasov-Shatashvili (NS) limit [12]

\[
G_{M}^{(k,...,k)} = \lim_{\epsilon_{1,2} \rightarrow 0} e_{1,2} G_{M}^{(k,...,k)}(\tau, m, \epsilon_\pm),
\]

which is of weight zero under modular transformations. This is in particular also the case for \( G_{M}^{(1,...,1)} \) which is crucial for constructing the NS limit \( \mathcal{F}_{N,M} \) of the free energy. The functions \( G_{M}^{(1,...,1)} \), which have weight zero and index \( MN \), capture the degeneracies of monopole strings realized as \( M \)-2-branes suspended between \( M5 \)-branes. There is evidence that they are the (equivariantly regularized) \( N = (2, 2) \) supersymmetric elliptic genera of the corresponding monopole moduli space \( Y_{M,N} \) of dimension \( 4MN \). In the limit \( m = \pm \frac{1}{2} e_1 e_2 \), the world-sheet supersymmetry of the \( M \)-strings enhances from (2,0) to (2,2) therefore in the NS limit at \( m = \frac{1}{2} \) the world-sheet supersymmetry is (2,2) for all \( M \). This enhancement of supersymmetry together with the weight and index of \( G_{M}^{(1,...,1)} \) suggests that \( G_{M}^{(1,...,1)} \) might be the elliptic genus of a sigma model with \( N = (2, 2) \) supersymmetry. This is supported by the fact that for \( M = 1 \) [4] \( \theta_- = \theta_1(\tau; m \pm \frac{i}{2}) \).
\[ \mathcal{F}_{N,M}^{NS} = \sum_{K \geq 1} Q^K \sum_{n=0}^{K-1} G^{(1,\ldots,1)}_{M,NS} \left( \frac{nt + i}{K/n}, nm, ne_1 \right) \]
\[ = \sum_{K \geq 1} Q^K \mathcal{H}_K \left( G^{(1,\ldots,1)}_{M,NS} \right). \quad (16) \]

Using \( \mathcal{F}_{N,M}^{NS} \), we can in turn define the partition function
\[ \hat{Z}_{N,M}^{NS} (\rho, \tau, m, e_1) = e^{\mathcal{F}_{NS}(\rho, \tau, m, e_1)} = \sum_{n \geq 0} Q^n \mathcal{Z}^{N,M} (\tau, m, e_1), \]
which only receives contributions from the BPS sector counted by \( \mathcal{F}_{N,M}^{NS} \). Here, we have
\[ \mathcal{Z}^{N,M} (\tau, m, e_1) = \frac{1}{n!} \sum_{|\lambda| = n} a_\lambda \mathcal{H}_\lambda \left( G^{(1,\ldots,1)}_{M,NS} \right), \quad (17) \]
where the sum is over all integer partitions \( \lambda \) of \( n \). Explicitly, for a partition \( \lambda = (1^{m_1} 2^{m_2} 3^{m_3} \ldots) \), we have
\[ a_\lambda = \frac{n!}{\prod m_i!}, \quad \mathcal{H}_\lambda = \prod \mathcal{H}_m^m. \quad (18) \]

Note that the conjugacy classes of \( S_n \) are equally labelled by integer partitions of \( n \) such that we can rewrite the relation (17) as a sum over conjugacy classes of \( S_n \). Furthermore, since \( \mathcal{Z}^{N,M} (\tau, m, e_1) \) is just \( G^{(1,\ldots,1)}_{M,NS} \), the elliptic genus of the target space \( Y_{N,M} \), the quantity \( \mathcal{Z}_n \) is indeed a conjugacy class of the orbifold target space \( Y_{N,M}^n / S_n \). The latter can be expressed in terms of sums over commuting pairs of elements of \( S_n \) which define the twisted boundary conditions on the torus:
\[ \mathcal{Z}_n^{N,M} = \frac{1}{n!} \sum_{(g,h) \in S_n} g \bigcirc h. \quad (19) \]

Here, \( g \bigcirc h = \mathcal{Z}^{N,M}_1 (\tau, g h, m, e_1) \), while \( g \bigcirc h \) can be written as [13]:
\[ g \bigcirc h = \prod_{(g,h) \in S_n} 3^{N,M} (\tau, g h, m, e_1) \quad \text{with} \quad g h = h g. \]

In this expression, \( g \bigcirc h \subset S_n \) is the subgroup of \( S_n \) generated by the (commuting pair of) elements \( g, h \) and \( \mathcal{O}(g, h) \) refers to the collection of orbits of \( g \bigcirc h \) when acting on the set \( \{1, \ldots, n\} \). Then, \( \tau \xi = \frac{\tau h \tau h}{2} \) (and \( \tau \xi \) is uniquely determined from the Hecke structure (17)), where \( \lambda_\xi \) is the length of any \( g \)-orbit in \( \xi \), \( \mu_\xi \) is the number of \( g \)-orbits in \( \xi \) (such that \( \lambda_\xi \mu_\xi = |\xi| \)) and \( k_1 \) is the minimal non-negative integer such that \( k_1 \xi = x^\xi \).
BOUND STATES OF LITTLE STRINGS AND SYMMETRIC …


PHYSICAL REVIEW D 96, 081901(R) (2017)