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Bound states of little strings and symmetric orbifold conformal field theories

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We study BPS bound states of little strings in a limit where they realize monopole strings in five dimensional gauge theories. The latter have gauge group $U(M)^N$ and arise from compactification of (1,0) little string theories of type $A_{M-1} \times A_{N-1}$. We find evidence that the partition function of a certain subclass of monopole strings of charge $(k, \ldots, k)$ $(k \geq 1)$ is expressible as the partition function of a symmetric orbifold sigma model, whose target space is precisely the symmetric product of the moduli space of monopoles with charge $(1, \ldots, 1)$.

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Little string theories (LSTs) refer to a quantum theory of noncritical strings without gravity, living in six spacetime dimensions. It has been noted that their properties are similar to those of strings encountered in other physical applications, such as hadronic strings in QCD and magnetic flux tubes in superconductors. It has also been noted that little string theories provide microstate descriptions of various supersymmetric black holes in four dimensions. When compactified to five dimensions, the theory gives rise to monopole strings [1–3], whose various properties can be studied using the underlying little strings [4]. Monopole strings, as the name indicates, are stringlike solutions in five dimensions which appear pointlike and carry monopole charge in the transverse three dimensions [5,6]. All these connections motivate a deeper study of the structure of LSTs.

LST of type $A_{N-1} \times A_{M-1}$ with $\mathcal{N} = (1, 0)$ supersymmetry can be engineered using N M5-branes probing a transverse A_{M-1} orbifold geometry. The M-theory background is given by $\mathbb{R}^4 \times \mathbb{T}^2 \times \mathbb{S}^1 \times \mathbb{R}^4/\mathbb{Z}_M$, where the M5-branes are extended along $\mathbb{R}^4 \times \mathbb{T}^2$ and separated along $\mathbb{S}^1$. In a series of papers [4,7–9], such LSTs of type A_{N-1} probing a $\mathbb{Z}_M$ orbifold background were studied and their partition functions were calculated by using a dual setup of D5- and NS5-branes in type IIB string theory. The latter in turn is dual to a particular class of toric Calabi-Yau threefolds $X_{N,M} \sim X_{1,1}/\mathbb{Z}_M \times \mathbb{Z}_N$ where $X_{1,1}$ resembles the resolved conifold near certain boundaries of the moduli space. The partition function of the little strings is refined by the insertion of three $U(1)$ currents: $U(1)_m$ corresponding to a rotation of the transverse $\mathbb{R}^4$ [8] and $U(1)_{\epsilon_{1,2}}$ acting on $\mathbb{R}^4 \sim \mathbb{C}^2$ as $(z_1, z_2) \mapsto (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$. When one of the world-volume directions is compactified, the corresponding five-dimensional world-volume theory on the M5-branes becomes a gauge theory with gauge group $U(M)^N$ broken down to $U(1)^{NM}$. The five-dimensional theory on $\mathbb{R}^3 \times \mathbb{T}^2$ contains monopole strings coming from M2-branes stretched between the M5-branes [2,3] and wrapped on $\mathbb{T}^2$. Thus, the partition function of little strings calculates the $\mathcal{N} = (2, 0)$ elliptic genus of monopole strings, in the limit $\epsilon_2 \mapsto 0$ which is required by the compactification of the corresponding direction on which $U(1)_{\epsilon_2}$ acts. These monopole strings carry charges $k_i | i = 1, \ldots, N$ (which is the number of M2-branes between the i-th and $i+1$-th M5-brane) as well as fractional momenta $p_a | a = 1, \ldots, M$ along the transverse $\mathbb{S}^1$.

In this paper, we study a subsector of the BPS spectrum consisting of states such that $k_1 = \ldots = k_N = k$ and $p_1 = \ldots = p_M = p$. Namely, this subsector with charges $(k, \ldots, k)$ contains k M2-branes starting and ending on any given M5-brane. We show that the degeneracies of these BPS states with quantum numbers (k, p) only depend on the product kp and therefore degeneracies of higher charge monopole strings of this type are completely determined by degeneracies of those with charge one. We further argue that the partition function of these monopole strings can be expressed in terms of a conformal field theory (CFT) whose target space is a symmetric orbifold.

The partition function of this class of LSTs, which we denote by $Z_{N,M}$, is given by [3,9,10]

$$Z_{N,M} = (W(\mathbf{T}, m, \epsilon_\pm))^N \mathcal{Z}_{N,M}(\mathbf{t}, \mathbf{T}, m, \epsilon_\pm), \qquad (1)$$

where $\mathbf{t} = (t_1, \ldots, t_N)$ with t_i (with $i = 1, \ldots, N$) being the separation between the i-th and the $(i+1)$-th M5-brane and $\mathbf{T} = (T_1, \ldots, T_M)$ determine the charge of the states with respect to the $\mathbb{Z}_M$ orbifold. Furthermore, m and $\epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}$ correspond to the $U(1)'s$ we discussed above, which are required to render $Z_{N,M}$ well-defined.

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The partition function (1) receives contributions from two parts. The first part $W(\mathbf{T}, m, \epsilon_\pm)$ is the contribution of a single M5-brane wrapped on a circle with transverse $\mathbb{Z}_M$ orbifold. This contribution can be determined by studying the modes of a single six-dimensional abelian tensor multiplet on a circle colored by the transverse orbifold. The second factor $\mathcal{Z}_{N,M}$ is the contribution coming from M2-branes suspended between the M5-branes. Since these M2-branes are the little strings in six dimensions, this second factor contains the little string and monopole string degeneracies and will be the object of study in this paper. It can be expanded in the following fashion

$$\mathcal{Z}_{N,M}(\mathbf{t}, \mathbf{T}, m, \epsilon_\pm) = \sum_{k_1 \ldots k_N} Q_1^{k_1} \ldots Q_N^{k_N} \mathcal{Z}_{N,M}^{k_1 \cdots k_N},$$

where $Q_i = e^{-t_i}$ and $\mathcal{Z}_{N,M}^{k_1 \cdots k_N}(\mathbf{T}, m, \epsilon_{1,2})$ captures the BPS states of k_i M2-branes suspended between the i-th and the $(i+1)$-th M5-brane, respectively.

The free BPS energy associated with the partition function (1) is given by $F_{N,M} = \ln(\mathcal{Z}_{N,M})$. In this paper, however, we study ($\bar{Q}_a = e^{-T_a}$)

$$\mathcal{F}_{N,M} = \sum_{k \geq 1, n \geq 0} Q_\rho^k Q_\tau^n \prod_{i=1}^{N} \prod_{a=1}^{M} \oint \frac{dQ_i d\bar{Q}_a}{Q_i^{k+1} \bar{Q}_a^{n+1}} \ln \mathcal{Z}_{N,M}, \qquad (2)$$

which only counts contributions from the aforementioned subspace of the BPS Hilbert space in which every M5-brane has the same number of M2-branes starting and ending on it. This reduced free energy $\mathcal{F}_{N,M}$ can be written as the infinite sum

$$\mathcal{F}_{N,M}(\rho, \tau, m, \epsilon_{1,2}) = \sum_{n=1}^{\infty} \frac{1}{n} G_{N,M}(n\rho, n\tau, nm, n\epsilon_\pm)$$

and only depends on $(\tau, \rho, m, \epsilon_\pm)$, which are sums of all $\mathbf{T}$ and $\mathbf{t}$, respectively:

$$\tau = \frac{i}{2\pi}(T_1 + \cdots + T_M) \quad \text{and} \quad \rho = \frac{i}{2\pi}(t_1 + \cdots + t_N).$$

Furthermore, $G_{N,M}(\rho, \tau, m, \epsilon_\pm) = \sum_{k \geq 1} Q_\rho^k G_M^{(k,\ldots,k)}$ (where the superscript contains N entries of k) captures the BPS degeneracies of k M2-branes stretched between any adjacent pair of the N M5-branes and winding k times around the transverse $\mathbb{S}^1$. These M2-branes carry arbitrary momenta along the $\mathbb{S}^1$ in M5-brane world volume.

For $N = M = 1$, the free energy $\mathcal{F}_{1,1}$ captures all the BPS states in the theory

$$\mathcal{F}_{1,1}(\tau, \rho, m, \epsilon_\pm) = \sum_{k>0} Q_\rho^k G_1^{(k)}(\tau, m, \epsilon_\pm), \qquad (3)$$

where $G_1^{(k)}$ can be expressed in terms of the contribution of a single M2-brane winding once around $\mathbb{S}^1$

$$G_1^{(k)} = \frac{1}{k} \sum_{a=0}^{k-1} G_1^{(1)}\left(\frac{\tau + a}{k}, m, \epsilon_\pm\right). \qquad (4)$$

Here, $G_1^{(1)}$ can be written as the following quotient of Jacobi theta-functions

$$G_1^{(1)}(\tau, m, \epsilon_\pm) = \frac{\theta_1(\tau; m + \epsilon_-)\theta_1(\tau; m - \epsilon_-)}{\theta_1(\tau; \epsilon_+ + \epsilon_-)\theta_1(\tau; \epsilon_+ - \epsilon_-)}. \qquad (5)$$

We see from the Fourier expansion of $G_1^{(k)}$

$$G_1^{(k)} = \sum_{n,\ell,r,s} c_k(n, \ell, r, s) e^{2\pi i \tau n} Q_m^\ell q^r t^s \qquad (6)$$

(where $q = e^{i\epsilon_1}$ and $t = e^{-i\epsilon_2}$) that relation (4) implies

$$c_k(n, \ell, r, s) = c_1(kn, \ell, r, s). \qquad (7)$$

In fact, the relation (4) reflects that $\mathcal{Z}_{1,1}$ is the partition function of a two-dimensional $\mathcal{N} = (2,2)$ supersymmetric sigma model whose target space is a symmetric product; *i.e.*, it is the partition function of a symmetric orbifold theory and therefore it can be expressed as [11]

$$\begin{aligned}\mathcal{Z}_{1,1} &= \prod_{k,n,\ell,r,s} (1 - Q_\rho^k Q_\tau^n Q_m^\ell q^r t^s)^{-c_1(kn,\ell,r,s)} \\ &= \sum_{k=0}^{\infty} Q_\rho^k \mathfrak{Z}((\mathbb{C}^2)^k / S_k),\end{aligned} \qquad (8)$$

where $\mathfrak{Z}((\mathbb{C}^2)^k / S_k)$ is the (equivariantly regularized) elliptic genus of $(\mathbb{C}^2)^k / S_k$ which can be written as

$$\mathfrak{Z}((\mathbb{C}^2)^k / S_k) = \frac{1}{k!} \sum_{\substack{hg = gh \\ g, h \in S_k}} g \underset{h}{\square}. \qquad (9)$$

In the sum, $g \underset{h}{\square}$ denotes the $\mathcal{N} = (2,2)$ partition function of a sigma model whose target space is the product of k copies of $\mathbb{C}^2$ and whose world-sheet boundary conditions in the space and time directions are twisted by h and g, respectively. It follows from Eqs. (3) and (4) that

$$\ln(\mathcal{Z}_{1,1}) = \sum_{k \geq 1} Q_\rho^k \mathcal{H}_k(G_1^{(1)}). \qquad (10)$$

Here, $\mathcal{H}_k$ denotes the kth Hecke operator: if $f_{w,\vec{r}}(\tau, \vec{z})$ is a Jacobi form of weight w and index $\vec{r}$ with respect to the (multi-)argument $\vec{z}$, then the kth Hecke transformation of $f_{w,\vec{r}}$ is defined as ($k \in \mathbb{N}$)

$$\mathcal{H}_k(f_{w,\vec{r}}(\tau, \vec{z})) = k^{w-1} \sum_{\substack{d|k \\ b \bmod d}} d^{-w} f_{w,\vec{r}}\left(\frac{k\tau + bd}{d^2}, \frac{k\vec{z}}{d}\right).$$

The function $\mathcal{H}_k(f_{w,\vec{r}}(\tau,\vec{z}))$ in turn is a Jacobi form of index $r\vec{k}$ and weight w. Notice that, under $SL(2,\mathbb{Z})$ transformations, $G_1^{(1)}$ transforms as

$$G_1^{(1)}\left(-\frac{1}{\tau},\frac{m}{\tau},\frac{\epsilon_\pm}{\tau}\right)=e^{\frac{2\pi i}{\tau}(m^2-\epsilon_+^2)}G_1^{(1)}(\tau,m,\epsilon_\pm);$$

i.e., it has weight $w=0$ and index $\vec{k}=(1,-1,0)$ with respect to $(m,\epsilon_+,\epsilon_-)$.

With Eqs. (8), (9) and (10), we have three representations of $\mathcal{Z}_{1,1}$ all of which follow from each other and are a consequence of the relation (4). In the following, we shall find a generalization of the latter for the particular BPS subsector contributing to $\mathcal{F}_{N,M}$ (2) in more complicated M-brane configurations: indeed, for arbitrary (N,M), a specific part of the partition function can be identified with the elliptic genus of a product of instanton moduli spaces and can again be written as the partition function of a symmetric orbifold theory. The free energy $\mathcal{F}_{N,M}$ (2) is given in terms of $G_M^{(k,\ldots,k)}$ by

$$\begin{aligned}\mathcal{F}_{N,M}&=\sum_{n\geq 1}\frac{1}{n}\sum_{k\geq 1}Q_\rho^{nk}G_M^{(k,\ldots,k)}(n\tau,nm,n\epsilon_\pm)\\&=\sum_{K\geq 1}Q_\rho^K\sum_{n|K}\frac{1}{n}G_M^{(K/n,\ldots,K/n)}(n\tau,nm,n\epsilon_\pm).\end{aligned}\tag{11}$$

The functions $G_M^{(k,\ldots,k)}(\tau,m,\epsilon_\pm)$ diverge in the limit $\epsilon_{1,2}\mapsto 0$ proportional to $\frac{1}{\epsilon_1\epsilon_2}$, and so we can define the slightly modified Nekrasov-Shatashvili (NS) limit [12]

$$G_{M,\mathrm{NS}}^{(k,\ldots,k)}=\lim_{\epsilon_2\mapsto 0}\frac{\epsilon_2}{\epsilon_1}G_M^{(k,\ldots,k)}(\tau,m,\epsilon_\pm),\tag{12}$$

which is of weight zero under modular transformations. This is in particular also the case for $G_{M,\mathrm{NS}}^{(1,\ldots,1)}$ which is crucial for constructing the NS limit $\mathcal{F}_{N,M}^{\mathrm{NS}}$ of the free energy. The functions $G_{M,\mathrm{NS}}^{(1,\ldots,1)}$, which have weight zero and index MN, capture the degeneracies of monopole strings realized as M2-branes suspended between M5-branes. There is evidence that they are the (equivariantly regularized) $\mathcal{N}=(2,2)$ supersymmetric elliptic genera of the corresponding monopole moduli space $Y_{M,N}$ of dimension $4MN$. In the limit $m=\pm\frac{\epsilon_1-\epsilon_2}{2}$, the world-sheet supersymmetry of the M-strings enhances from (2,0) to (2,2) therefore in the NS limit at $m=\pm\frac{\epsilon_1}{2}$ the world-sheet supersymmetry is (2,2) for all M. This enhancement of supersymmetry together with the weight and index of $G_{M,NS}^{(1,\ldots,1)}$ suggests that $G_{M,NS}^{(1,\ldots,1)}$ might be the elliptic genus of a sigma model with $\mathcal{N}=(2,2)$ supersymmetry. This is supported by the fact that for $M=1$ [4] $(\theta_\pm=\theta_1(\tau;m\pm\frac{\epsilon_1}{2}))$

$$G_{1,\mathrm{NS}}^{(1,\ldots,1)}=NG_{1,\mathrm{NS}}^{(1)}\left(\frac{\theta_+\theta_-'-\theta_-\theta_+'}{\theta_1(\tau;\epsilon_1)\eta^3(\tau)}\right)^{N-1}\tag{13}$$

is reduced to a constant N for $m=\pm\frac{\epsilon_1}{2}$, as expected of the $\mathcal{N}=(2,2)$ supersymmetric elliptic genus. Hereafter, we provide evidence that, in the limit (12), the degeneracies of bound states of k M2-branes organize themselves as

$$G_{M,\mathrm{NS}}^{(k,\ldots,k)}(\tau,m,\epsilon_1)=\frac{1}{k}\sum_{i=0}^{k-1}G_{M,\mathrm{NS}}^{(1,\ldots,1)}\left(\frac{\tau+i}{k},m,\epsilon_1\right).\tag{14}$$

M2-brane configurations in the NS limit are identified as monopole strings in five dimensions, so the relation (14), which constitutes the main result of this paper, is the statement that the degeneracies of bound states of monopole strings of charge $(k,\ldots,k)$ are completely determined by the degeneracies of charge $(1,\ldots,1)$ monopole strings [11]. The relation (14) can equally be written in terms of Hecke transformations (also clarifying the modular transformation properties of $G_{M,\mathrm{NS}}^{(k,\ldots,k)}$)

$$G_{M,\mathrm{NS}}^{(k,\ldots,k)}=\mathcal{T}_k(G_{M,\mathrm{NS}}^{(1,\ldots,1)}),\tag{15}$$

where the operator $\mathcal{T}_k$ acts in the following manner on a Jacobi form $f_{w,\vec{r}}(\tau,\vec{z})$ of weight w and index $\vec{r}$

$$\mathcal{T}_k(f_{w,r}(\tau,\vec{z}))\coloneqq\sum_{a|k}a^{w-1}\mu(a)\mathcal{H}_{\frac{k}{a}}(f_{w,\vec{r}}(a\tau,a\vec{z})),$$

and $\mu(a)$ is the Möbius function. We can also characterize Eq. (15) in a different fashion: If we take the Fourier expansion of $G_{M,\mathrm{NS}}^{(1,\ldots,1)}$ as

$$G_{M,\mathrm{NS}}^{(1,\ldots,1)}(\tau,m,\epsilon_1)=\sum_{p=0}^{\infty}\sum_{n\in\mathbb{Z}}c_{N,M}(p,n,r)Q_\tau^pQ_m^nq^r,$$

we have the following expansion for $G_{M,\mathrm{NS}}^{(k,\ldots,k)}$

$$G_{M,\mathrm{NS}}^{(k,\ldots,k)}=\mathcal{T}_k(G_{M,\mathrm{NS}}^{(1,\ldots,1)})=\sum_{p=0}^{\infty}\sum_{n\in\mathbb{Z}}c_{N,M}(kp,n,r)Q_\tau^pQ_m^nq^r.$$

Using the relation (14), we can express the NS limit of the reduced free energy $\mathcal{F}_{N,M}$ (2) as

$$\begin{aligned}\mathcal{F}_{N,M}^{\mathrm{NS}}&=\lim_{\epsilon_2\mapsto 0}\frac{\epsilon_2}{\epsilon_1}\mathcal{F}_{N,M}(\rho,\tau,m,\epsilon_1,\epsilon_2)\\&=\sum_{K\geq 1}Q_\rho^K\sum_{n|K}\frac{1}{n}G_{M,\mathrm{NS}}^{(K/n,\ldots,K/n)}(n\tau,nm,n\epsilon_1),\end{aligned}$$

which with the relation (14) allows us to write $\mathcal{F}_{N,M}^{\mathrm{NS}}$ as a sum over Hecke transformations of $G_{M,\mathrm{NS}}^{(1,\ldots,1)}$:

$$\mathcal{F}^{\rm NS}_{N,M} = \sum_{K\geq 1} Q_\rho^K \sum_{n|K} \frac{1}{K} \sum_{i=0}^{K/n-1} G^{(1,\ldots,1)}_{M,\rm NS}\left(\frac{n\tau+i}{K/n}, nm, n\epsilon_1\right) = \sum_{K\geq 1} Q_\rho^K \mathcal{H}_K(G^{(1,\ldots,1)}_{M,\rm NS}). \tag{16}$$

Using $\mathcal{F}^{\rm NS}_{N,M}$, we can in turn define the partition function

$$\tilde{\mathcal{Z}}^{NS}_{N,M}(\rho,\tau,m,\epsilon_1) = e^{\mathcal{F}_{NS}(\rho,\tau,m,\epsilon_1)} = \sum_{n\geq 0} Q_\rho^n \mathfrak{Z}_n^{N,M}(\tau,m,\epsilon_1),$$

which only receives contributions from the BPS sector counted by $\mathcal{F}^{\rm NS}_{N,M}$. Here, we have

$$\mathfrak{Z}_n^{N,M}(\tau,m,\epsilon_1) = \frac{1}{n!}\sum_{|\lambda|=n} a_\lambda \mathcal{H}_\lambda(G^{(1,\ldots,1)}_{M,\rm NS}), \tag{17}$$

where the sum is over all integer partitions λ of n. Explicitly, for a partition $\lambda = (1^{m_1}2^{m_2}3^{m_3}\cdots)$, we have

$$a_\lambda = \frac{n!}{\prod_i m_i!}, \qquad \mathcal{H}_\lambda = \prod_i \mathcal{H}_i^{m_i}. \tag{18}$$

Note that the conjugacy classes of S_n are equally labelled by integer partitions of n such that we can rewrite the relation (17) as a sum over conjugacy classes of S_n. Furthermore, since $\mathfrak{Z}_1^{N,M}(\tau,m,\epsilon_1)$ is just $G^{(1,\ldots,1)}_{M,\rm NS}$, the elliptic genus of the target space $Y_{N,M}$, the quantity $\mathfrak{Z}_n$ is the elliptic genus of the orbifold target space $(Y_{N,M})^n/S_n$. The latter can be expressed in terms of sums over commuting pairs of elements of S_n which define the twisted boundary conditions on the torus:

$$\mathfrak{Z}_n^{N,M} = \frac{1}{n!}\sum_{\substack{gh=hg\\ g,h\in S_n}} g\underset{h}{\square}. \tag{19}$$

Here, $e\underset{e}{\square} = \mathfrak{Z}_1^{N,M}$, while $g\underset{h}{\square}$ can be written as [13]:

$$g\underset{h}{\square} = \prod_{\xi\in\mathcal{O}(g,h)} \mathfrak{Z}_1^{N,M}(\tau_\xi, r_\xi m, r_\xi\epsilon_1) \quad \text{with} \quad \begin{matrix} g,h\in S_n, \\ gh=hg. \end{matrix}$$

In this expression, $\mathfrak{s}(g,h)\subset S_n$ is the subgroup of S_n generated by the (commuting pair of) elements g, h and $\mathcal{O}(g,h)$ refers to the collection of orbits of $\mathfrak{s}(g,h)$ when acting on the set $\{1,\ldots,n\}$. Then, $\tau_\xi = \frac{\mu_\xi\tau+\kappa_\xi}{\lambda_\xi}$ (and r_ξ is uniquely determined from the Hecke structure (17)), where λ_ξ is the length of any g-orbit in ξ, μ_ξ is the number of g-orbits in ξ (such that $\lambda_\xi\mu_\xi = |\xi|$) and κ_ξ is the minimal non-negative integer such that $h^{\mu_\xi} = x^{\kappa_\xi}$.

TABLE I. Orders to which Eq. (4) has been checked through explicit series expansion.

expression	order in Q_τ	order in ϵ_1
$G^{(2)}_{1,\rm NS} = \mathcal{T}_2(G^{(1)}_{1,\rm NS})$	20	5
$G^{(3)}_{1,\rm NS} = \mathcal{T}_3(G^{(1)}_{1,\rm NS})$	10	3
$G^{(4)}_{1,\rm NS} = \mathcal{T}_4(G^{(1)}_{1,\rm NS})$	5	3
$G^{(2,2)}_{1,\rm NS} = \mathcal{T}_2(G^{(1,1)}_{1,\rm NS})$	4	all orders
$G^{(3,3)}_{1,\rm NS} = \mathcal{T}_3(G^{(1,1)}_{1,\rm NS})$	2	all orders
$G^{(2,2,2)}_{1,\rm NS} = \mathcal{T}_2(G^{(1,1,1)}_{1,\rm NS})$	1	all orders
$G^{(2,2)}_{2,\rm NS} = \mathcal{T}_2(G^{(1)}_{2,\rm NS})$	3	all orders
$G^{(2,2)}_{3,\rm NS} = \mathcal{T}_2(G^{(1)}_{3,\rm NS})$	3	all orders

It remains to provide evidence for relation (14) which is at the heart of reduced free energy (16). We have checked the former equation by considering explicit series expansions in the parameters Q_τ and ϵ_1 and comparing both sides as a function of Q_m. The order of Q_τ and ϵ to which we checked these relations is tabulated in Table I. This indeed provides strong evidence for relation (14); we conjecture that it holds in full generality.

To conclude, we find that the representation (17) shows that the reduced partition function $\tilde{\mathcal{Z}}^{\rm NS}_{N,M}$ is expressible as the partition function of a symmetric orbifold CFT. Using the duality proposed in [4], we can further interpret $\tilde{\mathcal{Z}}^{\rm NS}_{N,M}$ as (part of) the BPS partition function of monopole strings compactified on $\mathbb{S}^1$. Thus, our results indicate that a BPS subsector of five-dimensional monopole strings can be written as a symmetric orbifold CFT. This CFT is a sigma model whose target space is the symmetric product of $Y_{N,M}$, where $Y_{N,M}$ is the moduli space of monopole strings with charge $(1,\ldots,1)$. Notice in this respect that, both for the interpretation of $\tilde{\mathcal{Z}}^{\rm NS}_{N,M}$ in terms of the monopole strings (see [4]) as well as its rewriting in terms of a symmetric orbifold partition function, the NS limit is crucial. We expect a similar orbifold description of monopole strings in little strings of D and E type as well.

One interesting application of our result is the counting of black hole microstates through the Ooguri-Strominger-Vafa (OSV) conjecture [14]. The little string partition function we studied is also the partition function of a subsector of the topological string on $X_{M,N}$. This implies that the black holes engineered from $X_{M,N}$ by the OSV conjecture have partition functions which in some cases can be expressed as the partition function of a sigma model. Our result indicates that perhaps in some limit this sigma model is reduced to a symmetric orbifold CFT.

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