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Master formulas for the dressed scalar propagator in a constant field

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Abstract

The worldline formalism has previously been used for deriving compact master formulas for the one-loop N-photon amplitudes in both scalar and spinor QED, and in the vacuum as well as in a constant external field. For scalar QED, there is also an analogous master formula for the propagator dressed with N photons in the vacuum. Here, we extend this master formula to include a constant field. The two-photon case is worked out explicitly, yielding an integral representation for the Compton scattering cross section in the field suitable for numerical integration in the full range of electric and magnetic field strengths.

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1. Introduction

The one-loop effective action in scalar QED has the well-known "worldline" or "Feynman–Schwinger" representation [1],

$$\Gamma[A] = -\int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int_P Dx(\tau)\ \mathrm{e}^{-\int_0^T d\tau [\frac{1}{4}\dot{x}^2 + ie\dot{x}^\mu A_\mu(x(\tau))]}\,. \tag{1}$$

Here m and T denote the mass and proper-time of the loop scalar, and $\int_P Dx(\tau)$ the path integral over closed loops in (euclidean) spacetime with periodicity T in the proper-time.

Strassler in 1992 [2] showed how to convert this path integral into the following master formula for the N-photon amplitudes:

$$\Gamma(k_1, \varepsilon_1; \dots; k_N, \varepsilon_N) = -(-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}}\, \mathrm{e}^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times \exp\Big\{ \sum_{i,j=1}^N \Big[\frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \Big] \Big\} \Big|_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_N}\,. \tag{2}$$

Here $G_B, \dot{G}_B, \ddot{G}_B$ are the "bosonic" worldline Green's function and its first and second derivatives,

$$\begin{aligned} G_B(\tau,\tau') &\equiv \left|\tau - \tau'\right| - \frac{(\tau-\tau')^2}{T}\,, \\ \dot{G}_B(\tau,\tau') &= \mathrm{sign}(\tau-\tau') - 2\frac{\tau-\tau'}{T}\,, \\ \ddot{G}_B(\tau,\tau') &= 2\delta(\tau-\tau') - \frac{2}{T}\,. \end{aligned} \tag{3}$$

Here a 'dot' always means a derivative with respect to the first variable, and we abbreviate $G_B(\tau_i,\tau_j) \equiv G_{Bij}$ etc. $G_B(\tau,\tau')$ is the Green's function for the second derivative operator $\frac{d^2}{d\tau^2}$ adapted to the periodicity, as well as to the "string-inspired" ('SI') boundary conditions

$$\int_0^T d\tau\, G_B(\tau,\tau') = \int_0^T d\tau'\, G_B(\tau,\tau') = 0\,, \tag{4}$$

(up to an irrelevant constant that has been omitted). Note that $\ddot{G}_B(\tau,\tau')$ contains a delta function that brings together two photon legs; this is how the seagull vertex arises in the worldline formalism.

The notation $\big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}$ means that the exponential should be expanded, and only the terms linear in each of the polarization vectors be kept. The photons are ingoing and still off-shell, so that these vectors are just book-keeping devices at this stage.

Originally, the same master formula (2) was derived by Bern and Kosower [3,4] from string theory as a generating expression from which to construct the one-loop on-shell N gluon amplitudes by way of a certain set of rules. It contains the information on the N-photon amplitudes in a form that is not only extremely compact, but also well-organized with respect to gauge invariance, particularly when combined with a certain integration-by-parts procedure [3,4,2,5]. Moreover, it combines into one integral the various Feynman diagrams differing by the ordering of the N photons. This may not seem very relevant at the one-loop level, however when the N-photon amplitudes are used as building blocks for multiloop amplitudes it leads to highly nontrivial representations combining Feynman diagrams of different topologies [6,7] (see also [8]).

In [2] also a generalization to the spinor QED was given (see [7] for generalizations to more general field theories).

Shaisultanov [9] then generalized both the scalar and spinor QED master formulas to the case of QED in a constant external field $F_{\mu\nu}$. For the scalar case, this generalized master formula can be written as [10,7]

$$\Gamma(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) = -(-ie)^N (2\pi)^D \delta(\sum k_i) \times \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} \mathrm{e}^{-m^2 T} \det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right] \prod_{i=1}^N \int_0^T d\tau_i \times \exp\left\{\sum_{i,j=1}^N \left[\frac{1}{2} k_i\cdot \mathcal{G}_{Bij}\cdot k_j - i\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij}\cdot k_j + \frac{1}{2}\varepsilon_i \cdot \ddot{\mathcal{G}}_{Bij}\cdot \varepsilon_j\right]\right\}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}, \tag{5}$$

where we have introduced the abbreviation $\mathcal{Z} \equiv eFT$. This master formula differs from the vacuum one, Eq. (2), only by the additional determinant factor $\det^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]$, which represents the dependence of the free (photonless) path integral on the external field, and a change of the worldline Green's function G_B to a new one $\mathcal{G}_B$ that holds information on the external field,

$$\mathcal{G}_B(\tau_i,\tau_j) = \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{Bij}} + i\mathcal{Z}\dot{G}_{Bij} - 1\right). \tag{6}$$

This Green's function obeys the same SI boundary conditions as the vacuum one, (4).

The master formula (5) and its spinor QED generalization [9,10] are usually more efficient for the calculation of photonic processes in a constant field than the standard method based on Feynman diagrams. Its applications include the vacuum polarization in a constant field [11,12,7], photon splitting in a magnetic field [13,7], and the two-loop Euler–Heisenberg Lagrangian in an electric/magnetic field [10,14,15,7,16] as well as in a self-dual [17] background field. See [18–24] for extensions to gravity and Einstein–Maxwell theory.

Much less has been done for the analogous amplitudes involving an open line. For scalar QED in the vacuum, already in 1996 Daikouji et al. [25] obtained the following master formula representing the scalar tree-level propagator dressed with N photons (Fig. 1):

$$D^{pp'}(k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) = (-ie)^N (2\pi)^D \delta^D\Big(p+p'+\sum_{i=1}^N k_i\Big)\int_0^\infty dT\, \mathrm{e}^{-(m^2+p^2)T} \times \prod_{i=1}^N \int_0^T d\tau_i\, \mathrm{e}^{\sum_{i,j=1}^N [\Delta_{ij} k_i\cdot k_j - 2i\,{}^\bullet\!\Delta_{ij}\varepsilon_i\cdot k_j - {}^\bullet\!\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_i]}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}. \tag{7}$$

Here a different worldline Green's function $\Delta(\tau,\tau')$ appears,

$$\Delta(\tau,\tau') = \frac{\tau\tau'}{T} + \frac{|\tau-\tau'|}{2} - \frac{\tau+\tau'}{2}. \tag{8}$$

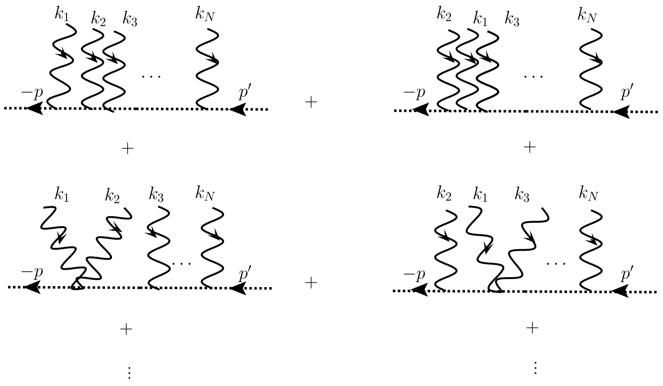

Fig. 1. Multi-photon Compton-scattering diagram.

Instead of the SI boundary conditions (4), it is adapted to Dirichlet boundary conditions ('DBC')

$$\Delta(0,\tau') = \Delta(T,\tau') = \Delta(\tau,0) = \Delta(\tau,T) = 0\,. \tag{9}$$

Contrary to the former, these boundary conditions break the translation invariance in proper-time, so that one now has to distinguish between derivatives with respect to the first and the second argument. A convenient notation is [26] to use left and right dots to indicate derivatives with respect to the first and the second argument, respectively:

$$\begin{aligned}
{}^{\bullet}\Delta(\tau_1,\tau_2) &= \frac{\tau_2}{T} + \frac{1}{2}\mathrm{sign}(\tau_1-\tau_2) - \frac{1}{2}\,,\\
\Delta^{\bullet}(\tau_1,\tau_2) &= \frac{\tau_1}{T} - \frac{1}{2}\mathrm{sign}(\tau_1-\tau_2) - \frac{1}{2}\,,\\
{}^{\bullet}\Delta^{\bullet}(\tau_1,\tau_2) &= \frac{1}{T} - \delta(\tau_1-\tau_2)\,.
\end{aligned} \tag{10}$$

The Green's functions G_B and Δ are related by [27]

$$G_B(\tau,\tau') = 2\Delta(\tau,\tau') - \Delta(\tau,\tau) - \Delta(\tau',\tau')\,, \tag{11}$$

(the factor of 2 is conventional) with the inverse relation

$$2\Delta(\tau,\tau') = G_B(\tau,\tau') - G_B(\tau,0) - G_B(0,\tau')\,. \tag{12}$$

In [25] the master formula (7) was derived by a comparison with the standard Schwinger parameter integral representations of the corresponding Feynman diagrams. Recently, the same formula has been rederived [28] inside the worldline formalism, starting from the generalization of the path integral representation (1) to the propagator of a scalar particle in the Maxwell background:

$$D^{xx'}[A] = \int_0^\infty dT\, \mathrm{e}^{-m^2T} \int_{x(0)=x'}^{x(T)=x} Dx\, \mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x)\right]}\,. \tag{13}$$

The master formula (7) so far has been generalized neither to spinor QED, nor to the inclusion of an external field. The purpose of the present paper is to carry out the latter generalization; the extension to the fermionic case (but without an external field yet) will be presented in a companion paper [29]. See [30] for a non-abelian generalization of the dressed scalar propagator.

The organization of the paper is as follows. As a warm-up, in section 2 we use the path integral representation to rederive the well-known scalar propagator in a constant field, in configuration as well as in momentum space. In section 3 we obtain our master formulas for the photon-dressed propagator in a constant field in both configuration and momentum space, generalizing the vacuum calculation of [28]. In section 4 we work the momentum space formula out for the $N=2$ case, and obtain a compact integral representation for the Compton scattering cross section in a constant field. Section 5 provides a summary and outlook. In Appendix A we give our conventions, while in Appendix B we collect some information on the constant field worldline Green's functions.

2. The propagator in a constant field

In this section, we use (13) to just rederive the well-known scalar propagator in a constant field, without photons yet.

2.1. Configuration space

Choosing Fock–Schwinger gauge, the gauge potential for a constant field can be written as

$$A^{\mu}(y)=-\frac{1}{2}F^{\mu\nu}(y-x')^{\nu}, \tag{14}$$

where we have fixed the initial point of the trajectory x' as the reference point where the potential will vanish.

Further, we decompose the arbitrary trajectory $x(\tau)$ into a straight-line part and a fluctuation part $q(\tau)$ obeying Dirichlet boundary conditions, $q(0)=q(T)=0$:

$$x(\tau)=x'+\frac{\tau}{T}(x-x')+q(\tau). \tag{15}$$

Using (14) and (15) in (13), and further defining

$$Q^{\mu}\equiv\int_0^T d\tau q^{\mu}(\tau)\,, \tag{16}$$

after some simple manipulations the integrand can be rewritten as

$$\begin{aligned}D^{xx'}(F)&=\int_0^\infty dT\,\mathrm{e}^{-m^2T}\,\mathrm{e}^{-\frac{(x-x')^2}{4T}}\int Dq(\tau)\,\mathrm{e}^{-\int_0^T d\tau\frac{\dot q^2}{4}+\frac{ie}{2}\int_0^T d\tau\dot q^{\mu}F_{\mu\nu}q^{\nu}+\frac{ie}{T}(x-x')^{\mu}F_{\mu\nu}Q^{\nu}}\\&=\int_0^\infty dT\,\mathrm{e}^{-m^2T}\,\mathrm{e}^{-\frac{(x-x')^2}{4T}}\int Dq(\tau)\,\mathrm{e}^{-\int_0^T d\tau\frac14 q\left(-\frac{d^2}{d\tau^2}+2ieF\frac{d}{d\tau}\right)q+\frac{ie}{T}(x-x')FQ}\,.\end{aligned} \tag{17}$$

The path integral is already of gaussian form, and in the second line we have written it in a form that prepares the formal gaussian integration. Apart from the free path integral normalization, which is (see, e.g., [31])

$$\int Dq(\tau)\,\mathrm{e}^{-\int_0^T d\tau q\left(-\frac14\frac{d^2}{d\tau^2}\right)q}=(4\pi T)^{-\frac{D}{2}}\,, \tag{18}$$

this involves the determinant and the inverse of the operator $-\frac{d^2}{d\tau^2}+2ieF\frac{d}{d\tau}$. For the case of the SI boundary conditions (4), the relevant formulas have been given already in (5), (6) above. The ratio of the field-dependent and free path integral normalizations are

$$\frac{\mathrm{Det}'^{-\frac{1}{2}}_P\left(-\frac{1}{4}\frac{d^2}{d\tau^2}+\frac{1}{2}ieF\frac{d}{d\tau}\right)}{\mathrm{Det}'^{-\frac{1}{2}}_P\left(-\frac{1}{4}\frac{d^2}{d\tau^2}\right)}=\mathrm{Det}'^{-\frac{1}{2}}_P\Big(\mathbb{1}-2ieF\big(\frac{d}{d\tau}\big)^{-1}\Big)=\det{}^{-\frac{1}{2}}\Big[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Big], \tag{19}$$

(the 'prime' refers to the elimination of the zero mode which is contained in the path integral for string-inspired boundary conditions). This can be shown by a direct eigenvalue computation [31], and it is easy to see that the spectrum does not change when passing from string-inspired to Dirichlet boundary conditions, so that (19) holds unchanged for the open-line case.

The worldline Green's function does change, but still relates to the one for string-inspired boundary conditions in the same way as in the vacuum case, (12):

$$\begin{aligned}\underset{\smile}{\Delta}(\tau,\tau')&\equiv\langle\tau|\Big(\frac{d^2}{d\tau^2}-2ieF\frac{d}{d\tau}\Big)^{-1}|\tau'\rangle_{\mathrm{DBC}}\\&=\frac{1}{2}\Big(\mathcal{G}_B(\tau,\tau')-\mathcal{G}_B(\tau,0)-\mathcal{G}_B(0,\tau')+\mathcal{G}_B(0,0)\Big).\end{aligned} \tag{20}$$

Note that, contrary to the vacuum Green's function (8), it is a non-trivial matrix in the Lorentz space–time indices. Using this Green's function in the usual completing-the-square procedure, we get

$$\begin{aligned}D^{xx'}(F)&=\int_0^\infty dT\,\mathrm{e}^{-m^2T}\,\mathrm{e}^{-\frac{(x-x')^2}{4T}}(4\pi T)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\left[\frac{\sin(eFT)}{eFT}\right]\\&\quad\times\exp\left\{\int_0^T d\tau\int_0^T d\tau'\frac{ie}{T}(x-x')F\underset{\smile}{\Delta}(\tau,\tau')\frac{ie}{T}F(x-x')\right\}\\&=\int_0^\infty dT\,\mathrm{e}^{-m^2T}\,\mathrm{e}^{-\frac{(x-x')^2}{4T}}(4\pi T)^{-\frac{D}{2}}\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\\&\quad\times\exp\left\{-\frac{1}{T^4}(x-x')\mathcal{Z}{}^\circ\underset{\smile}{\Delta}{}^\circ\mathcal{Z}(x-x')\right\}.\end{aligned} \tag{21}$$

Here we have now extended the above 'dot' notation to include integration as well as differentiation; a left (right) 'open circle' on $\underset{\smile}{\Delta}(\tau,\tau')$ denotes an integral $\int_0^T d\tau$ ($\int_0^T d\tau'$). In Appendix B we show that

$${}^\circ\underset{\smile}{\Delta}{}^\circ\equiv\int_0^T d\tau\int_0^T d\tau'\underset{\smile}{\Delta}(\tau,\tau')=\frac{T^3}{4\mathcal{Z}}\Big(\cot\mathcal{Z}-\frac{1}{\mathcal{Z}}\Big), \tag{22}$$

which brings us to the well-known proper-time representation of the constant-field propagator (see, e.g., [32]),

$$D^{xx'}(F)=\int_0^\infty dT\,\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\exp\left\{-\frac{1}{4T}(x-x')\mathcal{Z}\cot\mathcal{Z}(x-x')\right\}. \tag{23}$$

2.2. *Momentum space*

We Fourier transform (23),

$$D^{pp'}(F) = \int d^D x \int d^D x' \, \mathrm{e}^{ip\cdot x + ip'\cdot x'} D^{xx'}(F) \,, \tag{24}$$

and changing the integration variables to

$$x_+ = \frac{1}{2}(x + x'), \quad x_- = x - x' \,, \tag{25}$$

we get

$$D^{pp'}(F) = \int_0^\infty dT \, \mathrm{e}^{-m^2 T} (4\pi T)^{-\frac{D}{2}} \det{}^{\frac{1}{2}} \left[\frac{\mathcal{Z}}{\sin \mathcal{Z}} \right] \times \int d^D x_- \int d^D x_+ \, \mathrm{e}^{\frac{i}{2}(p-p')\cdot x_- + i(p+p')x_+} \exp\left\{ -\frac{1}{4T} x_- \mathcal{Z} \cot \mathcal{Z} x_- \right\} . \tag{26}$$

As

$$\int d^D x_- \, \mathrm{e}^{ip\cdot x_- - \frac{1}{4T} x_- \mathcal{Z} \cot \mathcal{Z} x_-} = \frac{(4\pi T)^{\frac{D}{2}}}{\det^{\frac{1}{2}} [\mathcal{Z} \cot \mathcal{Z}]} \mathrm{e}^{-Tp(\mathcal{Z} \cot \mathcal{Z})^{-1} p} \,, \tag{27}$$

the final result becomes

$$D^{pp'}(F) = (2\pi)^D \delta(p + p') D(p, F) \,,$$
$$D(p, F) = \int_0^\infty dT \; \mathrm{e}^{-m^2 T} \, \frac{\mathrm{e}^{-Tp(\frac{\tan \mathcal{Z}}{\mathcal{Z}})p}}{\det^{\frac{1}{2}} [\cos \mathcal{Z}]} \,. \tag{28}$$

3. The dressed propagator in a constant field

We now wish to dress the propagator with N photons in addition to the constant field. As before, we start in configuration space.

3.1. *Configuration space*

For this purpose, the potential in (13) has to be chosen as

$$A = A_{\mathrm{ext}} + A_{\mathrm{phot}} \,, \tag{29}$$

where A_{ext} is the same as in (14), and A_{phot} represents a sum of plane waves:

$$A^\mu_{\mathrm{phot}}(x) = \sum_{i=1}^N \varepsilon_i^\mu \; \mathrm{e}^{ik_i \cdot x} \,. \tag{30}$$

Each photon then effectively gets represented by a vertex operator

$$V^A[k, \varepsilon] = \int_0^T d\tau \, \varepsilon \cdot \dot{x}(\tau) \, \mathrm{e}^{ik\cdot x(\tau)} \,, \tag{31}$$

integrated along the scalar line. This leads to the following path integral representation of the constant-field propagator dressed with N photons:

$$D^{xx'}(F\,|\,k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) = (-ie)^N \int_0^\infty dT\, \mathrm{e}^{-m^2T} \int_P Dx\, \mathrm{e}^{-\int_0^T d\tau [\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A_{\rm ext}(x)]} \times V[k_1,\varepsilon_1]V[k_2,\varepsilon_2]\cdots V[k_N,\varepsilon_N]\,. \tag{32}$$

For the evaluation of the path integral, it will be convenient to rewrite the photon vertex operator (31) as

$$V^A[k,\varepsilon] = \int_0^T d\tau\, \mathrm{e}^{ik\cdot x(\tau)+\varepsilon\cdot\dot{x}(\tau)}\Big|_{\mathrm{lin}(\varepsilon)}\,. \tag{33}$$

Applying the path decomposition (15) we get the following generalization of (17),

$$\begin{aligned} &D^{xx'}(F\,|\,k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) \\ &= (-ie)^N \int_0^\infty dT \mathrm{e}^{-m^2T-\frac{x_-^2}{4T}} \int Dq\, \mathrm{e}^{-\int_0^T d\tau \frac{1}{4} q\left(-\frac{d^2}{d\tau^2}+2ieF\frac{d}{d\tau}\right)q + \frac{ie}{T}x_- FQ} \\ &\times \int_0^T \prod_{i=1}^N d\tau_i\, \mathrm{e}^{\sum_{i=1}^N \left(\varepsilon_i\cdot\frac{x_-}{T}+\varepsilon_i\cdot\dot{q}(\tau_i)+ik_i\cdot x_-\frac{\tau_i}{T}+ik_i\cdot x'+ik_i\cdot q(\tau_i)\right)}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}\,. \end{aligned} \tag{34}$$

The path integral is already in a form suitable for gaussian integration. "Completing the square" using the Green's function (20), and using (22), we get the following x-space master formula:

$$\begin{aligned} &D^{xx'}(F\,|\,k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) \\ &= (-ie)^N \int_0^\infty dT\, \mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}} \det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \mathrm{e}^{-\frac{1}{4T}x_-\mathcal{Z}\cot\mathcal{Z}x_-} \\ &\times \int_0^T d\tau_1 \cdots \int_0^T d\tau_N\, \mathrm{e}^{\sum_{i=1}^N\left(\varepsilon_i\cdot\frac{x_-}{T}+ik_i\cdot\frac{x_-\tau_i}{T}+ik_i\cdot x'\right)} \\ &\times \exp\Big[\sum_{i,j=1}^N\Big(k_i\,\underset{\smile}{\Delta}_{ij}\,k_j - 2i\varepsilon_i\,{}^\bullet\underset{\smile}{\Delta}_{ij}\,k_j - \varepsilon_i\,{}^\bullet\underset{\smile}{\Delta}{}^\bullet_{ij}\,\varepsilon_j\Big) \\ &+ \frac{2e}{T}x_-\sum_{i=1}^N\Big(F\,{}^\circ\underset{\smile}{\Delta}_i\,k_i - iF\,{}^\circ\underset{\smile}{\Delta}{}^\bullet_i\,\varepsilon_i\Big)\Big]\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}\,. \end{aligned} \tag{35}$$

For the special case of a purely magnetic field, this x-space master formula was obtained already in 1994 by McKeon and Sherry [33].

3.2. Momentum space

The transition to momentum space is quite analogous to the photon-less case. We Fourier transform according to (24), and change the variables to (25). The x_+ integral produces the global delta function for energy–momentum conservation, and the x_- integral is gaussian. Performing it we get our momentum space master formula:

$$
\begin{aligned}
&D^{pp'}(F \mid k_1, \varepsilon_1; \cdots; k_N, \varepsilon_N) \\
&= (-ie)^N (2\pi)^D \delta\Big(p + p' + \sum_{i=1}^N k_i\Big) \int_0^\infty dT\, \mathrm{e}^{-m^2 T} \frac{1}{\det^{\frac{1}{2}}[\cos\mathcal{Z}]} \\
&\times \int_0^T d\tau_1 \cdots \int_0^T d\tau_N\, \mathrm{e}^{\sum_{i,j=1}^N (k_i \underset{\smile}{\Delta}_{ij} k_j - 2i\varepsilon_i {}^\bullet\underset{\smile}{\Delta}_{ij} k_j - \varepsilon_i {}^\bullet\underset{\smile}{\Delta}{}^\bullet_{ij} \varepsilon_j)}\, \mathrm{e}^{-Tb(\frac{\tan\mathcal{Z}}{\mathcal{Z}})b}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}.
\end{aligned}
\tag{36}
$$

Here we have defined

$$
b \equiv p + \frac{1}{T}\sum_{i=1}^N \Big[\Big(\tau_i - 2ieF\, {}^\circ\underset{\smile}{\Delta}_i\Big)k_i - i\Big(1 - 2ieF\, {}^\circ\underset{\smile}{\Delta}{}^\bullet_i\Big)\varepsilon_i\Big]. \tag{37}
$$

The master formula (36) describes the same set of Feynman diagrams depicted in Fig. 1, only that now all the scalar propagators are the "full" ones in the external field (usually indicated by a double line). When applying it to the calculation of physical processes, one has to take into account that it describes the *untruncated* dressed propagator, i.e. the final propagators on each end of the scalar line in Fig. 1 are included. To obtain the matrix element $\mathcal{T}$, we have to cancel these final propagators using (28):

$$
\mathcal{T}(F \mid k_1, \varepsilon_1; \cdots; k_N, \varepsilon_N) = \frac{D^{pp'}(F \mid k_1, \varepsilon_1; \cdots; k_N, \varepsilon_N)}{D(p, F)D(p', F)}. \tag{38}
$$

Moreover, it will be convenient to Wick rotate from euclidean to Minkowski space; the rules for the Wick rotation are given in Appendix A together with our conventions. In Appendix B we collect the formulas necessary to write the integrand in explicit form. We use (20) to write the Green's function $\underset{\smile}{\Delta}(\tau, \tau')$ in terms of $\mathcal{G}_B(\tau, \tau')$, which is translation invariant and obeys (4) which will be very useful here. We then explain how to write $\mathcal{G}_B(\tau, \tau')$ explicitly for a generic constant field.

Finally, let us remark that eventual poles in the global proper-time integral due to the determinant factor in (36) are spurious, because when $\cos\mathcal{Z} = 0$ the factor $\mathrm{e}^{-Tb(\frac{\tan\mathcal{Z}}{\mathcal{Z}})b}$ will vanish too (differently from the corresponding one-loop amplitudes, where such poles lead to an imaginary part related to pair creation).

4. Compton scattering in a constant field

We will now work out the $N = 2$ case, i.e. Compton scattering in a constant field. Expanding out the exponentials in (36) and projecting to the terms linear in both polarization vectors, we find (omitting now the global factor for energy–momentum conservation):

$$
\begin{aligned}
D^{pp'}(F \mid k_1, \varepsilon_1; k_2, \varepsilon_2) &= e^2 \int_0^\infty dT \frac{\mathrm{e}^{-m^2 T}}{\det^{\frac{1}{2}}[\cos\mathcal{Z}]} \\
&\times \int_0^T d\tau_1 \int_0^T d\tau_2\, \mathrm{e}^{-Tb_0(\frac{\tan\mathcal{Z}}{\mathcal{Z}})b_0 + \sum_{i,j=1}^2 k_i \underset{\smile}{\Delta}_{ij} k_j}\, \varepsilon_1 M_{12} \varepsilon_2,
\end{aligned}
\tag{39}
$$

with

$$b_0 \equiv p + \frac{1}{T}\sum_{i=1}^{2}\Big(\tau_i - 2ieF\,{}^{\circ}\underset{\smile}{\Delta}_i\Big)k_i\,, \tag{40}$$

and

$$\begin{aligned} M_{12} \equiv\ & 2\,{}^{\bullet}\underset{\smile}{\Delta}{}^{\bullet}{}_{12} - \frac{2}{T}\Big(1 + 2ie\,{}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_1 F\Big)\frac{\tan\mathcal{Z}}{\mathcal{Z}}\Big(1 - 2ieF\,{}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_2\Big) \\ & + 4\Big[(1 + 2ie\,{}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_1 F)\frac{\tan\mathcal{Z}}{\mathcal{Z}}b_0 - \sum_{i=1}^{2}{}^{\bullet}\underset{\smile}{\Delta}_{1i}\,k_i\Big] \\ & \times\Big[b_0\frac{\tan\mathcal{Z}}{\mathcal{Z}}\Big(1 - 2ieF\,{}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_2\Big) - \sum_{i=1}^{2}k_i\underset{\smile}{\Delta}{}^{\bullet}{}_{i2}\Big]. \end{aligned} \tag{41}$$

Squaring, and performing the sum over the photon polarizations via

$$\sum_{\text{pol}}\varepsilon_i^{*\mu}\varepsilon_i^{\nu} \longrightarrow g^{\mu\nu}\,, \tag{42}$$

we get the following for the Compton cross section:

$$\begin{aligned} \sum_{\text{pol}}\mathcal{T}^*\mathcal{T} =\ & \frac{e^4}{|D(p,F)|^2\,|D(p',F)|^2} \\ & \times\int_0^\infty dT'\,\frac{\mathrm{e}^{-m^2T'}}{\det^{\frac{1}{2}}\left[\cos\mathcal{Z}'\right]}\int_0^{T'}d\tau_1'\int_0^{T'}d\tau_2'\,\mathrm{e}^{-T'b_0^*(\frac{\tan\mathcal{Z}'}{\mathcal{Z}'})b_0^* + \sum_{i,j=1}^{2}k_i\,\underset{\smile}{\Delta}'_{ij}\,k_j} \\ & \times\int_0^\infty dT\,\frac{\mathrm{e}^{-m^2T}}{\det^{\frac{1}{2}}\left[\cos\mathcal{Z}\right]}\int_0^{T}d\tau_1\int_0^{T}d\tau_2\,\mathrm{e}^{-Tb_0(\frac{\tan\mathcal{Z}}{\mathcal{Z}})b_0 + \sum_{i,j=1}^{2}k_i\,\underset{\smile}{\Delta}_{ij}\,k_j}\,\mathrm{tr}(M_{12}'^{\dagger}M_{12})\,. \end{aligned} \tag{43}$$

After writing the integrand explicitly with the help of the formulas of Appendix B, this expression is suitable for numerical integration.

5. Summary and outlook

Using the worldline path integral formalism, we have derived a Bern–Kosower type master formula for the scalar propagator in QED, in a constant field and dressed by an arbitrary number of photons. The x-space version of this formula generalizes the one obtained by McKeon and Sherry for the purely magnetic case [33]; the p-space version generalizes the vacuum master formula of Daikouji et al. [25] on one hand, the closed-loop master formula of Shaisultanov [9] on the other. Our master formula is valid off-shell, and combines the various orderings of the N photons along the scalar line. It can thus be used as a convenient starting point for the construction of higher-loop scalar QED processes in a constant field. On-shell, it yields parameter integral representations for linear and nonlinear Compton scattering in the field, as well as the various processes related to it by crossing.

To make this paper self-contained, we have also provided all the machinery necessary for writing the integrands in explicit form. We have worked out the integrand for the linear Compton scattering case explicitly, arriving at a compact representation suitable for numerical integration. The results of such a numerical computation will be presented in a forthcoming publication. Compton scattering in magnetic fields is a process of potential relevance for astrophysics, but, to the best of our knowledge, so far has been studied only in the strong-field limit [35].

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Appendix A. Conventions

At the path integral level, we work in the Euclidean space with a positive definite metric $(g_{\mu\nu}) = \mathrm{diag}(++\ldots+)$. The euclidean field strength tensor is defined by $F^{ij} = \varepsilon_{ijk}B_k$, $i,j = 1,2,3$, $F^{4i} = -iE_i$. Minkowski space amplitudes are obtained by analytically continuing

$$\begin{aligned} g_{\mu\nu} &\to \eta_{\mu\nu}\,, \\ k^4 &\to -ik^0\,, \\ T &\to is\,, \\ F^{4i} &\to F^{0i} = E_i\,, \end{aligned} \tag{A.1}$$

where $(\eta_{\mu\nu}) = \mathrm{diag}(-+++)$. These Minkowski space conventions agree with [34] up to the sign of the charge e.

Momenta appearing in vertex operators are *ingoing*.

Appendix B. Worldline Green's functions

Here we collect the information necessary to work out explicitly the integrands generated by the master formulas (35) and (36) for any N.

B.1. Expressing the DBC Green's function through the SI one

Rather than writing out the DBC Green's function $\underset{\smile}{\Delta}(\tau,\tau')$ and its derivatives directly in terms of trigonometric functions of the field strength tensor, we find it convenient to first rewrite them in terms of the SI Green's function $\mathcal{G}_B(\tau,\tau')$ via (20),

$$\underset{\smile}{\Delta}(\tau,\tau') = \frac{1}{2}\Big(\mathcal{G}_B(\tau,\tau') - \mathcal{G}_B(\tau,0) - \mathcal{G}_B(0,\tau') + \mathcal{G}_B(0,0)\Big)\,. \tag{B.1}$$

The advantages of $\mathcal{G}_B(\tau,\tau')$ are that it is translation invariant, so that we do not have to distinguish between right and left derivatives, and that it fulfills the same nonlocal boundary conditions as the vacuum Green's function (4),

$$\int_0^T d\tau\, \mathcal{G}_B(\tau,\tau') = \int_0^T d\tau\, \dot{\mathcal{G}}_B(\tau,\tau') = \int_0^T d\tau\, \ddot{\mathcal{G}}_B(\tau,\tau') = 0\,. \tag{B.2}$$

The latter property will be very useful for the 'circled' Green's functions. Moreover, the Lorentz matrix structure of $\mathcal{G}_B(\tau,\tau')$ has already been worked out for the various types of constant fields [12,7].

Using (B.1), the various derivatives and integrals of $\underset{\smile}{\Delta}(\tau,\tau')$ appearing in the master formulas become

$$\begin{aligned}
{}^\bullet\underset{\smile}{\Delta}(\tau,\tau') &= \frac{1}{2}\Big(\dot{\mathcal{G}}_B(\tau,\tau') - \dot{\mathcal{G}}_B(\tau,0)\Big)\,,\\
{}^\bullet\underset{\smile}{\Delta}{}^\bullet(\tau,\tau') &= -\frac{1}{2}\ddot{\mathcal{G}}_B(\tau,\tau')\,,\\
{}^\circ\underset{\smile}{\Delta}(\tau') &= \frac{T}{2}\Big(-\mathcal{G}_B(0,\tau') + \mathcal{G}_B(0,0)\Big)\,,\\
{}^\circ\underset{\smile}{\Delta}{}^\bullet(\tau') &= \frac{T}{2}\dot{\mathcal{G}}_B(0,\tau')\,,\\
{}^\circ\underset{\smile}{\Delta}{}^\circ &= \frac{T^2}{2}\mathcal{G}_B(0,0)\,.
\end{aligned} \tag{B.3}$$

B.2. General properties of the Green's function $\mathcal{G}_B$

Here we cite a few general properties of the Green's function $\mathcal{G}_B$ and its derivatives; for derivations and more details, see [12,7,23]. We can write these functions as power series in the matrix $\mathcal{Z} \equiv eFT$ as follows [10]:

$$\begin{aligned}
\mathcal{G}_B(\tau,\tau') &= \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\,\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')} + i\mathcal{Z}\dot{G}_B(\tau,\tau') - 1\right),\\
\dot{\mathcal{G}}_B(\tau,\tau') &= \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')} - 1\right),\\
\ddot{\mathcal{G}}_B(\tau,\tau') &= 2\delta(\tau-\tau') - \frac{2}{T}\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')}\,.
\end{aligned} \tag{B.4}$$

By absorbing the dependence on τ,τ' in terms of the derivative of the vacuum Green's function, $\dot{G}_B(\tau,\tau')$, one avoids having to make an explicit case distinction between $\tau_1 > \tau_2$ and $\tau_1 < \tau_2$ that would become necessary otherwise [9]. Let us note also the coincidence limits of $\mathcal{G}_B, \dot{\mathcal{G}}_B$:

$$\begin{aligned}
\mathcal{G}_B(\tau,\tau) &= \frac{T}{2\mathcal{Z}^2}\Big(\mathcal{Z}\cot(\mathcal{Z}) - 1\Big)\,,\\
\dot{\mathcal{G}}_B(\tau,\tau) &= i\cot(\mathcal{Z}) - \frac{i}{\mathcal{Z}}\,.
\end{aligned} \tag{B.5}$$

Note that they are independent of τ. As Lorentz matrices, $\mathcal{G}_B$ and its derivatives have the following symmetry properties:

$$\mathcal{G}_B(\tau,\tau') = \mathcal{G}_B^T(\tau',\tau), \quad \dot{\mathcal{G}}_B(\tau,\tau') = -\dot{\mathcal{G}}_B^T(\tau',\tau), \quad \ddot{\mathcal{G}}_B(\tau,\tau') = \ddot{\mathcal{G}}_B^T(\tau',\tau)\,. \tag{B.6}$$

For weak background fields, it is often justified to approximate the Green's function by the first few terms of its expansion in $F_{\mu\nu}$. To order F^2, one finds

$$\begin{aligned}
\mathcal{G}_B(\tau,\tau') &= G_B(\tau,\tau') - \frac{T}{6} - \frac{i}{3}\dot{G}_B(\tau,\tau')G_B(\tau,\tau')TeF + \Big(\frac{T}{3}G_B^2(\tau,\tau') - \frac{T^3}{90}\Big)(eF)^2 \\
&\quad + O(F^3)\,, \\
\dot{\mathcal{G}}_B(\tau,\tau') &= \dot{G}_B(\tau,\tau') + 2i\Big(G_B(\tau,\tau') - \frac{T}{6}\Big)eF + \frac{2}{3}\dot{G}_B(\tau,\tau')G_B(\tau,\tau')T(eF)^2 \\
&\quad + O(F^3)\,, \\
\ddot{\mathcal{G}}_B(\tau,\tau') &= \ddot{G}_B(\tau,\tau') + 2i\dot{G}_B(\tau,\tau')eF - 4\Big(G_B(\tau,\tau') - \frac{T}{6}\Big)(eF)^2 + O(F^3)\,.
\end{aligned} \tag{B.7}$$

These expansions are easily obtained from (B.4) using the identity $\dot{G}_B^2(\tau,\tau') = 1 - \frac{4}{T}G_B(\tau,\tau')$. The coefficients can be written in closed form to all orders in F, either in terms of Bernoulli polynomials of $\tau - \tau'$ [10], or in terms of Faulhaber polynomials of $\dot{G}_B(\tau,\tau')$ [23].

B.3. Matrix decomposition of the Green's function $\mathcal{G}_B$

Finally, a matrix decomposition of $\mathcal{G}_B$ will be necessary. This can be achieved in a Lorentz invariant way [12], but from a practical point of view it is simpler to work in a Lorentz frame well-adapted to the external field. Here, we will be satisfied with treating (i) the case of a generic field and (ii) the purely magnetic field case; see [12] for the more special 'crossed field' and 'self-dual' cases. In all cases it will be useful to decompose $\mathcal{G}_B$ as

$$\mathcal{G}_B = \mathcal{S}_B + \mathcal{A}_B\,, \tag{B.8}$$

where $\mathcal{S}_B$ is the even part of $\mathcal{G}_B$ as a function of F, and $\mathcal{A}_B$ the odd one. For $\mathcal{S}_B$, the following trigonometric rewriting is often useful:

$$\mathcal{S}_B(\tau,\tau') - \mathcal{S}_B(\tau,\tau) = T\,\frac{\sin\big(|u-u'|\,\mathcal{Z}\big)\sin[(1-|u-u'|)\mathcal{Z}]}{\mathcal{Z}\sin\mathcal{Z}}\,, \tag{B.9}$$

where we have rescaled $\tau = Tu$, $\tau' = Tu'$.

B.3.1. The generic case

For a generic constant field, both Maxwell invariants $\mathbf{B}^2 - \mathbf{E}^2$ and $\mathbf{E}\cdot\mathbf{B}$ are nonzero. By Lorentz invariance there then exists a Lorentz frame where the electric and magnetic field vectors both point along the z-axis, and by parity invariance we can assume that they both point along the *positive* z-axis. In euclidean conventions, we then have

$$F = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & iE \\ 0 & 0 & -iE & 0 \end{pmatrix}, \tag{B.10}$$

which suggests to introduce the following matrix base:

$$g_\perp \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_\parallel \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_\perp \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; r_\parallel \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Using this Lorentz frame and base, and defining

$$z_\perp \equiv eBT, \quad z_\parallel \equiv ieET\,, \tag{B.11}$$

the matrix functions $\mathcal{S}_B$ and $\mathcal{A}_B$ can be decomposed as [12,7]

$$\begin{aligned}
\mathcal{S}^{\mu\nu}_{B12} &= -\frac{T}{2}\sum_{\alpha=\perp,\parallel}\frac{A^\alpha_{B12}}{z_\alpha}\,g^{\mu\nu}_\alpha\,,\\
\mathcal{A}^{\mu\nu}_{B12} &= \frac{iT}{2}\sum_{\alpha=\perp,\parallel}\frac{S^\alpha_{B12}-\dot G_{B12}}{z_\alpha}\,r^{\mu\nu}_\alpha\,,\\
\dot{\mathcal{S}}^{\mu\nu}_{B12} &= \sum_{\alpha=\perp,\parallel} S^\alpha_{B12}\,g^{\mu\nu}_\alpha\,,\\
\dot{\mathcal{A}}^{\mu\nu}_{B12} &= -i\sum_{\alpha=\perp,\parallel} A^\alpha_{B12}\,r^{\mu\nu}_\alpha\,,\\
\ddot{\mathcal{S}}^{\mu\nu}_{B12} &= \ddot G_{B12}g^{\mu\nu} - \frac{2}{T}\sum_{\alpha=\perp,\parallel} z_\alpha A^\alpha_{B12}\,g^{\mu\nu}_\alpha\,,\\
\ddot{\mathcal{A}}^{\mu\nu}_{B12} &= \frac{2i}{T}\sum_{\alpha=\perp,\parallel} z_\alpha S^\alpha_{B12}\,r^{\mu\nu}_\alpha\,,
\end{aligned} \tag{B.12}$$

with the following coefficient functions:

$$\begin{aligned}
S^\alpha_{B12} &= \frac{\sinh(z_\alpha \dot G_{B12})}{\sinh(z_\alpha)}\,,\\
A^\alpha_{B12} &= \frac{\cosh(z_\alpha \dot G_{B12})}{\sinh(z_\alpha)} - \frac{1}{z_\alpha}
\end{aligned} \tag{B.13}$$

($\alpha = \perp, \parallel$). In the worldline formalism, these two scalar, dimensionless functions S_B and A_B are the basic building blocks of the integrands of one-loop amplitudes in a constant field in scalar QED, as well as in scalar Einstein–Maxwell theory [21,23].

B.3.2. The magnetic case

For easy reference, let us write down here also the explicit formulas for the case of a pure magnetic field, with **B** pointing along the z-axis:

$$\begin{aligned}
\bar{\mathcal{G}}_{B12} &= G_{B12}\,g_\parallel - \frac{T}{2}\frac{\Big(\cosh(z\dot G_{B12}) - \cosh(z)\Big)}{z\sinh(z)}g_\perp\\
&\quad + \frac{T}{2z}\left(\frac{\sinh(z\dot G_{B12})}{\sinh(z)} - \dot G_{B12}\right)ir_\perp\,,\\
\dot{\mathcal{G}}_{B12} &= \dot G_{B12}\,g_\parallel + \frac{\sinh(z\dot G_{B12})}{\sinh(z)}g_\perp - \left(\frac{\cosh(z\dot G_{B12})}{\sinh(z)} - \frac{1}{z}\right)ir_\perp\,,
\end{aligned}$$

$$\ddot{\bar{\mathcal{G}}}_{B12} = \ddot{G}_{B12}\, g_{\parallel} + 2\Big(\delta_{12} - \frac{z\cosh(z\dot{G}_{B12})}{T\sinh(z)}\Big) g_{\perp} + 2\frac{z\sinh(z\dot{G}_{B12})}{T\sinh(z)} i r_{\perp}\,. \tag{B.14}$$

Here the "bar" on $\mathcal{G}_B$ indicates that its irrelevant coincidence limit has been subtracted. The DBC Green's function in the magnetic case can be written relatively compactly as [33]

$$\underset{\smile}{\Delta}(\tau,\tau') = \Delta(\tau,\tau')g_{\parallel} + \frac{2eT}{z}\Big[\theta(\tau-\tau')\sin\frac{z(\tau-\tau')}{2eT} - \frac{\sin\frac{z\tau}{2eT}\sin\frac{z(T-\tau')}{2eT}}{\sin\frac{z}{2e}}\Big]$$
$$\times\Big[\cos\frac{z(\tau-\tau')}{2eT} g_{\perp} + \sin\frac{z(\tau-\tau')}{2eT} r_{\perp}\Big], \tag{B.15}$$

where now $z = eBT$.

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