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Electric and magnetic energy at axion haloscopes

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We review the electro-magnetic energy at axion haloscopes and find that the electric and the corresponding magnetic energy stored in the cavity modes or, equivalently, the mode dependent electric and magnetic form factors are the same regardless of the position of the cavity inside the solenoid. Furthermore, we extend our argument to the cases satisfying $\vec{\nabla}\times\vec{B}_{\text{external}}=0$, where $\vec{B}_{\text{external}}$ is a static magnetic field provided by a magnet at an axion haloscope. Two typical magnets, solenoidal and toroidal, satisfy $\vec{\nabla}\times\vec{B}_{\text{external}}=0$; thus, the electric and the corresponding magnetic energy stored in the cavity modes are always the same in both cases. The energy, however, is independent of the position of the cavity in axion haloscopes with a solenoid, and depends on those with a toroidal magnet.

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I. INTRODUCTION

The axion is a hypothetical particle that was proposed as a solution to the strong *CP* problem [1–7]. This particle is massive, abundant, and very weakly interacting, making it a promising candidate for cold dark matter if the axion mass is above 1 $\mu\text{eV}/c^2$ [8–10] and below 3 meV/c^2 [11–15].

The axion search method proposed by Sikivie [16], also known as the axion haloscope search, involves a microwave resonant cavity with a strong static magnetic field that induces axion conversions into microwave photons, where the electromagnetic energy from the interaction inside the cavity is

$$U_{a,em}=g_{a\gamma\gamma}^2\langle a^2\rangle c^2\epsilon_0 B_{\text{avg}}^2 V C_{\text{mode}}, \tag{1}$$

where $g_{a\gamma\gamma}$ is the axion-photon coupling strength, $\langle a^2\rangle$ is the mean square of the spatially homogeneous axion field, $a=a_0e^{-i\omega_a t}$, with angular frequency ω_a, corresponding to the axion mass, and amplitude a_0 [17], c the speed of light (thus, $c^2\epsilon_0=1/\mu_0$). B_{avg}^2 is defined with $B_{\text{avg}}^2V\equiv\int\vec{B}_{\text{external}}^2dV$, where $\vec{B}_{\text{external}}$ is a static magnetic field provided by magnets at axion haloscopes, and V is the volume of the cavity. The form factor of the cavity depending on a resonant mode C_{mode} is

$$C_{\text{mode}}=\frac{(\int_V dV\sum_{i=1}^3 E_{i,\text{mode}}B_i)^2}{B_{\text{avg}}^2V\int_V dV\sum_{i=1}^3\sum_{j=1}^3 E_{i,\text{mode}}\epsilon_{ij}E_{j,\text{mode}}}, \tag{2}$$

where B_i is the component of $\vec{B}_{\text{external}}$, $E_{i,\text{mode}}$ is the electric field component of a resonant mode, and ϵ_{ij} is component of the relative dielectric tensor. In the axion haloscopes to date [18–22], Eq. (2) has been used to calculate the form factor of a cylindrical cavity that is centered in and occupies the complete volume of a solenoidal field.

Recently, a report [23] pointed out that Eq. (2) actually corresponds only to electric energy from axion to photon conversions inside the cavity $U_{a,e}$, thus they referred to Eq. (2) as the electric form factor C_E. The report also pointed out that an implicit assumption that $U_{a,e}$ and $U_{a,m}$ are the same has been made in most all axion haloscopes, where $U_{a,m}$ is the magnetic energy from the conversion inside the cavity. Hence the assumption results in $U_{a,em}=2U_{a,e}$, which, according to the report, has been valid to date for a cylinder in a solenoid as mentioned in the previous paragraph. Furthermore, the report introduced the magnetic form factor of the cavity modes C_B or, equivalently, the magnetic energy stored in the cavity modes of axion haloscopes and claimed that C_B depends on the offset of a cylindrical cavity from the center of the solenoid while the corresponding electric form factors do not show such dependence. This statement is effectively equivalent to that the electric energy stored in the cavity modes is generally different from the corresponding magnetic energy in axion haloscopes.

We find that as long as $\vec{\nabla}\times\vec{B}_{\text{external}}=0$ is valid, the electric and magnetic energy stored in the cavity modes are the same for both a solenoid and a toroidal magnet, which shows there is no such dependence with cylindrical axion haloscopes with solenoidal B fields as recognized in Ref. [24].

II. MAXWELL'S EQUATIONS WITH AXION MODIFICATIONS

Our review starts from Maxwell's equations with modification due to the axion field [25]

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$$\begin{aligned}
\vec{\nabla}\cdot(\vec{E}-cg_{a\gamma\gamma}a\vec{B}) &= \frac{\rho_e}{\epsilon_0},\\
\vec{\nabla}\cdot\left(\vec{B}+\frac{1}{c}g_{a\gamma\gamma}a\vec{E}\right) &= \mu_0\rho_m,\\
\vec{\nabla}\times(\vec{E}-cg_{a\gamma\gamma}a\vec{B}) &= -\frac{\partial}{\partial t}\left(\vec{B}+\frac{1}{c}g_{a\gamma\gamma}a\vec{E}\right)-\mu_0\vec{J}_m,\\
\vec{\nabla}\times\left(\vec{B}+\frac{1}{c}g_{a\gamma\gamma}a\vec{E}\right) &= \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{E}-cg_{a\gamma\gamma}a\vec{B})+\mu_0\vec{J}_e,
\end{aligned} \tag{3}$$

where ρ_e and $\vec{J}_e$ are the electric charge density and current, ρ_m and $\vec{J}_m$ are the magnetic charge density and current if magnetic monopoles exist.[1] In Eq. (3), $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields that are not induced from the interaction ruled by $g_{a\gamma\gamma}$, but participate in that interaction [26]. We will refer to the electric and magnetic fields induced from the coupling as $\vec{E}_a$ and $\vec{B}_a$ which are related to the second terms in parentheses in Eq. (3). In principle, $\vec{E}_a$ and $\vec{B}_a$ can also interact with the axion field to induce additional axion to photon conversions, but the interactions are proportional to $g^2_{a\gamma\gamma}a^2$ of which the magnitude is $\mathcal{O}(10^{-43})$, thus can be ignored in Eq. (3).

III. MAXWELL'S EQUATIONS WITH AXIONS ONLY

We make the following simplifying assumptions

(i) vacuum, thus $\rho_e=\rho_m=0$ and $\vec{J}_e=\vec{J}_m=\vec{0}$,

(ii) no electromagnetic radiation, thus $\vec{E}=0$, but $\vec{B}=\vec{B}_{\text{external}}$.

With our simplifying assumptions and ignoring terms with $g^2_{a\gamma\gamma}a^2\sim\mathcal{O}(10^{-43})$, $\vec{\nabla}\cdot\vec{E}$, $\vec{\nabla}\cdot\vec{B}$, and $\vec{\nabla}\times\vec{E}$ are zero, while $\vec{\nabla}\times\vec{B}=\vec{\nabla}\times\vec{B}_{\text{external}}$ is equated to

$$-\frac{1}{c}g_{a\gamma\gamma}\vec{B}_{\text{external}}\frac{\partial a}{\partial t}, \tag{4}$$

which can be found in elsewhere [16]. In the presence of a time-varying axion field, Eq. (4) is not zero, while $\vec{\nabla}\times\vec{B}_{\text{external}}$ is zero with ideal solenoids and toroidal magnets or, equivalently, $\vec{B}_{\text{external}}=B_0\hat{z}$ or $\frac{B_0}{\rho}\hat{\phi}$, where B_0 is a constant, z, ρ, and ϕ refer to cylindrical coordinates. Therefore, in order for the Maxwell's equations with axion modifications to be valid under solenoidal and toroidal magnetic fields, Eq. (4) should induce magnetic field $\vec{B}_a$ according to Ampere-Maxwell law. Since Eq. (4) can be regarded as a time-varying displacement current, $\mu_0\vec{J}_a$, induced from the variation of axion field or a time-varying electric field induced from the variation of the axion field, thus also can be equated to $\frac{1}{c^2}\frac{\partial\vec{E}_a}{\partial t}$, which results in

$$\begin{aligned}
\vec{\nabla}\times\vec{B}_a &= -\frac{1}{c}g_{a\gamma\gamma}\vec{B}_{\text{external}}\frac{\partial a}{\partial t}\\
&= \mu_0\vec{J}_a\\
&= \frac{1}{c^2}\frac{\partial\vec{E}_a}{\partial t}.
\end{aligned} \tag{5}$$

In the presence of $\vec{E}_a$ and $\vec{B}_a$ shown in Eq. (5), the modified Maxwell's equations satisfying our simplifying assumptions are

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E}_a &= 0,\\
\vec{\nabla}\cdot\vec{B}_a &= 0,\\
\vec{\nabla}\times\vec{E}_a &= -\frac{\partial\vec{B}_a}{\partial t},\\
\vec{\nabla}\times\vec{B}_a &= \frac{1}{c^2}\frac{\partial\vec{E}_a}{\partial t}.
\end{aligned} \tag{6}$$

Note that Eq. (6) is a result of the first order approximation of $g_{a\gamma\gamma}a$ in Eq. (3) and valid only if $\vec{\nabla}\times\vec{B}_{\text{external}}=0$, which is the case with axion haloscopes in solenoidal or toroidal magnetic fields. Therefore, Eq. (6) is quite different from the Maxwell's equation with electromagnetic radiation only, thus is the most important output in this paper. At the end, however, Eq. (6) is the same form as the Maxwell's equations with electromagnetic radiation only, meaning one can treat $\vec{E}_a$ and $\vec{B}_a$ in the same way that has always been done when treating the electric and magnetic fields associated with radiation.

At axion haloscopes, we are actually interested in the energy stored in the cavity modes which are the time-varying electric ($\vec{E}_{ac}$) and magnetic ($\vec{B}_{ac}$) fields induced by the axion field. Note that $\vec{E}_{ac}$ and $\vec{B}_{ac}$ will have to satisfy the boundary conditions for a given cavity at the end. Then, $\vec{E}_{ac}$ and $\vec{B}_{ac}$ must satisfy Maxwell's equations in Eq. (7) if $\vec{\nabla}\times\vec{B}_{\text{external}}=0$

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E}_{ac} &= 0,\\
\vec{\nabla}\cdot\vec{B}_{ac} &= 0,\\
\vec{\nabla}\times\vec{E}_{ac} &= -\frac{\partial\vec{B}_{ac}}{\partial t} = i\omega_a\vec{B}_{ac},\\
\vec{\nabla}\times\vec{B}_{ac} &= \frac{1}{c^2}\frac{\partial\vec{E}_{ac}}{\partial t} = -\frac{i\omega_a}{c^2}\vec{E}_{ac},
\end{aligned} \tag{7}$$

[1] No experimental evidence for magnetic monopole yet, thus, $\rho_m=0$ and $\vec{J}_m=0$ in Eq. (3) would be desirable.

under our simplifying assumptions, where the second equalities are from the oscillation of $\vec{E}_{ac}$ and $\vec{B}_{ac}$ with the frequency corresponding to the axion mass ω_a.

IV. ELECTRIC AND MAGNETIC ENERGY AT AXION HALOSCOPES

One can get the relation between electric and magnetic energy from Eq. (7)

$$\vec{E}_{ac} \cdot \vec{E}^*_{ac} = c^2 \vec{B}_{ac} \cdot \vec{B}^*_{ac} - \frac{ic^2}{\omega_a} \vec{\nabla} \cdot (\vec{E}^*_{ac} \times \vec{B}_{ac}). \quad (8)$$

The second term in right-hand side of Eq. (8) goes to zero with the boundary conditions at the cavity surface because we do not expect electromagnetic energy flow through the cavity surface in an ideal case. Therefore, Eq. (8) states that the electric energy stored in a cavity mode $U_{ac,e} \propto \vec{E}_{ac} \cdot \vec{E}^*_{ac}$ and the corresponding magnetic energy stored in a cavity mode $U_{ac,m} \propto \vec{B}_{ac} \cdot \vec{B}^*_{ac}$ are the same, regardless of the position of the cavity inside the magnet as long as $\vec{\nabla} \times \vec{B}_{\text{external}} = 0$. In the absence of dielectric material in the system, $U_{a,e}$ and $U_{a,m}$ can be equated to $U_{ac,e}$ and $U_{ac,m}$, respectively. Furthermore, because with $U_{a,e} \propto \vec{E}_a \cdot \vec{E}_{ac}$ ($U_{a,m} \propto \vec{B}_a \cdot \vec{B}_{ac}$) [23], one can also relate E_a to E_{ac} (B_a to B_{ac}) by equating $U_{a,e}$ and $U_{ac,e}$ ($U_{a,m}$ and $U_{ac,m}$). Since we show $U_{ac,e} = U_{ac,m}$ regardless of the position of the cavity inside the magnet as long as $\vec{\nabla} \times \vec{B}_{\text{external}} = 0$, $U_{a,e} = U_{a,m}$ is true; thus, $C_E = C_B$ is also true regardless of the position of the cavity inside the solenoid, which is different than the argument in Ref. [23].[2] Figure 1 shows the numerical demonstration that the magnetic energy from axion conversions into microwave photons inside a cylindrical cavity is independent of the position of the cavity inside the solenoid using the finite element method [27]. This numerical demonstration is calculated with the TM_{010} mode only. The three-dimensional data points of $\vec{B}_a$ and $\vec{B}_{ac}$ are simulated for the solenoidal geometry, then the magnetic energy from the conversion $U_{a,m}$ is calculated from $\int_V dV \vec{B}_a \cdot \vec{B}_{ac}$ as a function of offset over the cavity radius, where the offset is the distance between the solenoid center and the cavity center. The magnetic form factors given in Ref. [23] are also calculated numerically and superimposed on Fig. 1.

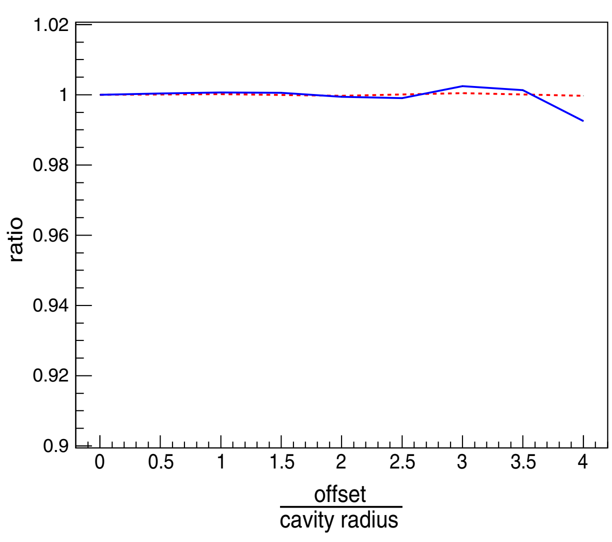

FIG. 1. Line (Dots) is ratios of $U_{a,m}$ (C_B) to $U^{\text{center}}_{a,m}$ (C^{center}_B) as a function of offset over the cavity radius, where $U^{\text{center}}_{a,m}$ (C^{center}_B) is the magnetic energy (magnetic form factor) when the cavity is located in the center of the solenoid and offset is the distance between the solenoid center and the cavity center. The results are calculated with the TM_{010} mode only.

V. ELECTRIC AND MAGNETIC ENERGY AT AXION HALOSCOPES WITH TOROIDAL GEOMETRY

As we pointed out earlier in this paper, the condition $\vec{\nabla} \times \vec{B}_{\text{external}} = 0$ is also true for ideal toroidal magnets. Therefore, $U_{a,e} = U_{a,m}$ regardless of the position of the toroidal cavity inside the toroidal magnet. This is very crucial in axion haloscopes with a toroidal magnet because one cannot define axion signal power without knowing $U_{a,e}$ and $U_{a,m}$ explicitly. The $U_{a,e}$ of toroidal system fortunately can be obtained analytically, but one cannot get $U_{a,m}$ of the system analytically, and thus cannot define the axion signal power from axion to microwave photon conversions with a toroidal magnet. However, since $U_{a,e} = U_{a,m}$ is also always true for a toroidal system, we do not have to know $U_{a,m}$ explicitly, instead we can double the $U_{a,e}$ to get the electromagnetic energy from axion to photon conversions inside the cavity $U_{a,em}$, which is what we experimentally measure at axion haloscopes.

$U_{a,e}$ and $U_{a,m}$ are proportional to $B^2_{\text{avg}} V$, therefore proportional to the magnetic energy inside the cavity. For a solenoid, $B^2_{\text{avg}} V$ is uniform regardless of the cavity location; therefore, $U_{a,e}$ and $U_{a,m}$ are uniform and independent of the position of the cavity inside the magnet. For a toroidal magnet, however, both B^2_{avg} and V vary depending on the cavity location; therefore, $U_{a,e}$ and $U_{a,m}$ depend on position of the cavity inside the magnet.

VI. SUMMARY

We have reviewed the electric and magnetic energy in axion haloscopes. Starting with Maxwell's equations modified due to the axion field, we find that the electric and

[2] We recently realized the authors of Ref. [23] submitted an erratum which can be found in arXiv:1607.01928. Nevertheless, our approach can be extended to any systems as long as $\vec{\nabla} \times \vec{B}_{\text{external}} = 0$ is valid.

corresponding magnetic energy stored in the cavity modes or, equivalently, the mode dependent electric and magnetic form factors are the same regardless of the position of the cavity inside a static magnetic field satisfying $\vec{\nabla} \times \vec{B}_{\mathrm{external}} = 0$. The energy, however, is independent of the position of the cavity in axion haloscopes with a solenoid, and depends on those with a toroidal magnet, due to with the $B_{\mathrm{avg}}^2 V$ dependence on it.

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