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Multiphoton amplitudes and generalized Landau-Khalatnikov-Fradkin transformation in scalar QED

Naser Ahmadiniaz,[1,2,4] Adnan Bashir,[1] and Christian Schubert[1,3]

[1]*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Apartado Postal 2-82, C.P. 58040, Morelia, Michoacan, Mexico*

[2]*Facultad de Ciencias en Física y Matemáticas, Universidad Autónoma de Chiapas, Ciudad Universitaria, Tuxtla Gutiérrez 29050, Tuxtla Gutiérrez, Mexico*

[3]*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Mühlenberg 1, D-14476 Potsdam, Germany*

[4]*Center for Relativistic Laser Science, Institute for Basic Science, Gwangju 61005, Korea*

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We apply the worldline formalism to amplitudes in scalar quantum electrodynamics involving open scalar lines, with an emphasis on their nonperturbative gauge dependence. At the tree level, we study the scalar propagator interacting with any number of photons in configuration space as well as in momentum space. At one loop we rederive, in an efficient way, the off-shell vertex in an arbitrary dimension and any covariant gauge. Generalizing the Landau-Khalatnikov-Fradkin transformation for the nonperturbative propagator, we find simple nonperturbative transformation rules for arbitrary x-space amplitudes under a change of the covariant gauge parameter in terms of conformal cross ratios.

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I. INTRODUCTION

Unraveling the nonperturbative structure of Green functions in gauge field theories has been a challenging task. A deep understanding of the emergent phenomena of confinement and dynamical chiral symmetry breaking in quantum chromodynamics (QCD) can be achieved only through the outcome of this endeavor. However, leaving aside the intricacies of a non-Abelian gauge field theory, there is a lot one can learn from the relatively simpler cases of spinor and scalar quantum electrodynamics (QED).

Perturbation theory and gauge covariance properties of Green functions have served as guiding principles toward our knowledge of their nonperturbative counterparts. Interaction vertices are naturally a focus of study in this context. A systematic study of the three-point electron-photon vertex in spinor QED was initiated more than three decades ago by Ball and Chiu [1]. They decompose the vertex into longitudinal and transverse parts, where the former satisfies the Ward-Fradkin-Green-Takahashi identity (WFGTI) [2]. They provide a set of eight basis vectors to write out the transverse vertex and calculate it at the one-loop level for off-shell external legs in four dimensions in the Feynman gauge. Their choice of the transverse basis guarantees that each of the corresponding coefficients is independent of any unwanted kinematic singularities.

Similarly, the one-loop electron-photon vertex in the Yennie-Fried gauge was evaluated in [3]. Later, the work of Ball and Chiu was extended to arbitrary covariant gauges by Kizilersü *et al.* [4]. However, this work shows that a slight change in the transverse basis is required to ensure the absence of kinematic singularities for each and every coefficient in an arbitrary covariant gauge. For the massive and massless three-point vertex in three-dimensional spinor QED (QED_3), the results in arbitrary covariant gauges were obtained in [5–7]. The superrenormalizability of QED_3 implies that the vertex has no ultraviolet divergences. QED_3 thus provides a neater laboratory to explore the effects of the electron-photon vertex on dynamical chiral symmetry breaking and confinement.

The QED vertex is useful not only by itself, but it also serves as a simple model for its subsequent extension to the more complicated non-Abelian case of QCD. Because of the identical Dirac nature of electrons and quarks, the quark-gluon vertex has the same number of basis vectors in its general decomposition, 12, as the electron-photon one.

The difference lies in the fact that it is now the Slavnov-Taylor identity (STI) [8] which extracts the longitudinal part, still leaving eight basis vectors for expanding the transverse vertex. There have been several works on the calculation of the quark-gluon vertex at one loop and beyond in different kinematic regimes of momenta [9], relating the symmetric point vertex to the running coupling in QCD. The systematic generalization of the QED three point vertex to the study of QCD can be attributed to Ball and Chiu [10]. Identically to the case of QED, they calculate the quark-gluon vertex to the one-loop order in the Feynman gauge and project it onto the basis they proposed earlier. Employing the modified basis of [4], Davydychev *et al.* [11] evaluate the one-loop quark-gluon vertex in an arbitrary gauge ξ and spacetime dimensions D in an $SU(N)$ gauge field theory. An appropriate choice of ξ, D and the color factors reproduces the results of earlier QED and QCD studies, just as one would expect. It is important to note that the knowledge of the three-point interactions in an arbitrary covariant gauge is a crucial guiding principle to pinpoint the transverse vertex, which remains undetermined through the WFGTI or STI. *Can the knowledge of a 3-point vertex in one gauge lead us to know what it is in any other covariant gauge without having to redo the calculation ab initio*? This is the kind of question we address in this article.

Our focus of attention is an even simpler gauge theory, namely scalar QED. Because of the absence of Dirac matrices, it only requires two basis vectors for its most general description. The longitudinal one is fixed by the WFGTI. The transverse part of the one-loop vertex was calculated by Ball and Chiu [1] in the Feynman gauge. However, the study of the gauge dependence of the Green functions and their connection to the lower point functions require their knowledge in an arbitrary covariant gauge. This study was carried out by Bashir *et al.* in [12,13]. In the present work, adopting the string-inspired worldline formalism [14–17], we set out to *construct this vertex in an arbitrary covariant gauge based on its knowledge solely in one gauge*.

On more general grounds, the gauge dependence of Green functions was formally studied by Landau and Khalatnikov [18], and independently by Fradkin [19]. They derived a series of transformations, dubbed as Landau-Khalatnikov-Fradkin transformations (LKFT), which encapsulate how the Green functions transform in a specific manner under a variation of gauge. The LKFT were later rederived by Johnson and Zumino through the use of functional methods [20]. These transformations are nonperturbative and written in coordinate space; obtaining an analytical counterpart for them in momentum space is not a simple task. The LKFT can not only be used to change from one covariant gauge to another at a fixed loop level but also to predict higher-loop terms from lower-loop ones. However, those predicted terms will all be gauge parameter dependent [21].

In the standard second-quantized formalism, the LKFT for the fermion propagator is relatively easier to investigate than for the scalar one. Studies in massless scalar and spinor QED, as well as in QCD (under certain conditions), demonstrate that the wave function renormalization has a multiplicatively renormalizable form of a power law in four dimensions [21,22]. In the quenched QED gap equation for the electron propagator, this solution can be reproduced only with an appropriate choice of the electron-photon three-point vertex. It is well known that the longitudinal Ball-Chiu vertex is not sufficient to ensure the LKFT law [23]. Since that realization, there have been a series of works which construct the electron-photon vertex implementing the multiplicative renormalizability of the electron propagator [23–26]. In [27], this LKFT law was implemented for the fermion propagator, and it simultaneously ensures the gauge invariance of the critical coupling above which chiral symmetry is dynamically broken. It also provides an infrared enhanced anomalous magnetic moment for quarks, advocated in [28].

All the above efforts are about constructing a three-point vertex which ensures the LKFT of the fermion propagator in its gap equation. However, the three-point vertices have their own transformation laws that were derived in [18–20]. *In this article, the local gauge transformation of the three-point scalar QED vertex is addressed through the worldline approach.*

In the present article, several of the above issues will be studied for the scalar QED case. We have three main motivations and objectives in mind: first, the multiphoton generalizations of Compton scattering are becoming important these days for laser physics; see, for example, [29] and [30]. Second, the computation of off-shell form factors for n-point Green functions in scalar QED can provide a simple yet nontrivial and insightful starting point to later go on to study spinor QED or QCD. And third, we look for efficient ways of transforming a Green function from one covariant gauge to another, independently for external and internal photons. In fact, the second and third objectives are interrelated as it is usually not sufficient to know form factors in just one gauge. In order to establish the gauge invariance of associated physical observables and to look for relations between different Green functions, their knowledge in different covariant gauges can be of valuable help. This observation is intricately related to the LKFT [18,19].

Our method of choice here is not the standard second-quantized formalism, but the string-inspired worldline formalism. Prior to embarking upon our main discussion, let us provide a brief summary of this not so well-known formalism (see [31,32] for reviews).

One of the main reasons for studying string theory is the fact that it provides us with an efficient mathematical framework that transcends quantum field theory, but reduces to it in the limit of infinite string tension. A systematic investigation of this limit led Bern and Kosower in 1991 [16] to a novel and efficient way to compute gauge theory amplitudes. In particular, they obtained a compact generating function for the one-loop (on-shell) N-gluon amplitudes, known as the *Bern-Kosower master formula*, and they applied it to a first calculation of the five-gluon amplitudes [33].

Later, Strassler [17] showed that many of their results can be obtained more straightforwardly using a representation of the S-matrix in terms of first-quantized particle path integrals, invented for the QED case by Feynman in 1950 [14,15]. This "string-inspired worldline formalism" is manifestly off-shell and uses string theory only as a guiding principle. In contrast to the standard Feynman diagrammatic approach, the main advantages which one can hope to obtain by employing this formalism are the following:

(1) Effectively the loop momenta have already been integrated out, which reduces the number of possible kinematic invariants from the beginning.

(2) In favorable cases, it allows one to derive compact master formulas that contain the information on large numbers of Feynman diagrams [32,34–38].
(3) In gauge theory, it is frequently possible to achieve manifest gauge invariance already *at the integrand level* [17,39–41].
(4) The method treats spin in a more uniform manner; in particular, calculations of amplitudes in spinor QED usually yield the corresponding results for scalar QED as a spin-off [17,39].
(5) It is generally easier than in the standard approach to incorporate constant external fields [42–46].

To give a recent example, two of the present authors have used the formalism to recalculate the off-shell three-gluon vertex [40], recuperating the form-factor decomposition originally proposed by Ball and Chiu [10], in a way that not only reduced the amount of algebra significantly, but also made the analysis of the Ward identities unnecessary. Moreover, it unified the scalar, spinor, and gluon loop cases. The superior efficiency of the method becomes even more conspicuous at the four-gluon level [41,47].

For scalar QED, as early as 1996 Daikouji *et al.* [36] showed how to apply the string-inspired worldline formalism to arbitrary amplitudes, albeit only in momentum space. However, for our present study of the nonperturbative gauge dependence, it will be essential to work in x-space as well as in momentum space.

The structure of this paper is as follows. In Secs. II and III we give short summaries of the worldline formalism and the LKFT, respectively. In Sec. IV, we derive our master formula for the scalar propagator dressed by an arbitrary number of photons, both in configuration and in momentum space. We then use this tree-level formula in Sec. V to construct, by sewing, the one-loop corrections to the scalar propagator and to the photon-scalar three-point vertex, in Feynman gauge and for arbitrary space-time dimension. In Sec. VI, we derive our central result, which is a generalization of the LKFT to arbitrary x-space amplitudes in scalar QED. The argument is based on the fact that changes of the covariant gauge parameter correspond to total derivative terms in the worldline integrands. This formula is nonperturbative. In Sec. VII, we take up an example to show how it works at the perturbative level. In Sec. VIII, we return to the one-loop propagator and vertex in momentum space and show how to obtain their form in an arbitrary covariant gauge from the one in Feynman gauge in an efficient way by identifying all difference terms as total derivative terms. We then compare our results with the ones obtained in [12] and observe complete agreement. In Sec. IX, we summarize our results and discuss possible generalizations. There are two appendixes: in Appendix A we give explicit results for some of the integrals appearing in the calculation of the one-loop propagator and vertex. Some details of the calculation of the vertex have been relegated to Appendix B.

We work with Euclidean conventions throughout.

II. THE WORLDLINE FORMALISM IN SCALAR QED

In this section, we discuss our method, which is based on the worldline formalism, developed by Feynman for scalar QED [14] and spinor QED [15], as an extension of his more standard path integral formalism for nonrelativistic quantum mechanics.

Although the formalism applies to arbitrary amplitudes in scalar QED, the essential features of the formalism can already be seen from the case of the quenched scalar propagator. Feynman's path integral representation [14] of the quenched propagator of a scalar particle of mass m, which propagates from point x' to x in the presence of a background $U(1)$ gauge field A, is

$$\Gamma[x,x'] = \int_0^\infty dT \mathrm{e}^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\mathrm{e}^{-S_0-S_e-S_i}, \tag{2.1}$$

where

$$\begin{aligned} S_0 &= \int_0^T d\tau \frac{1}{4}\dot{x}^2, \\ S_e &= ie\int_0^T d\tau\, \dot{x}\cdot A(x(\tau)), \\ S_i &= \frac{e^2}{2}\int_0^T d\tau_1 \int_0^T d\tau_2 \dot{x}_1^\mu D_{\mu\nu}(x_1-x_2)\dot{x}_2^\nu. \end{aligned} \tag{2.2}$$

Here, S_0 describes the free propagation, S_e the interaction of the scalar with the external field, and S_i virtual photons exchanged along the scalar's trajectory. We abbreviate $x(\tau_i) \equiv x_i$, etc. $D_{\mu\nu}$ is the x-space photon propagator in D dimensions. In an arbitrary covariant gauge, it is given by

$$\begin{aligned} D_{\mu\nu}(x) = \frac{1}{4\pi^{\frac{D}{2}}}\bigg\{&\frac{1+\xi}{2}\Gamma\left(\frac{D}{2}-1\right)\frac{\delta_{\mu\nu}}{(x^2)^{\frac{D}{2}-1}} \\ &+(1-\xi)\Gamma\left(\frac{D}{2}\right)\frac{x_\mu x_\nu}{(x^2)^{\frac{D}{2}}}\bigg\}. \end{aligned} \tag{2.3}$$

Here $\xi = 1$ corresponds to the Feynman gauge and $\xi = 0$ to the Landau gauge.

The expansion of the exponentials of the interaction terms S_e and S_i generates the Feynman diagrams depicted in Fig. 1. The external legs represent interactions with the field $A(x)$ and are converted into momentum-space photons by choosing $A(x)$ as a sum of plane waves,

$$A^\mu(x) = \sum_{i=1}^N \varepsilon_i^\mu\, \mathrm{e}^{ik_i\cdot x}. \tag{2.4}$$

Each external photon then gets represented by a *vertex operator*,

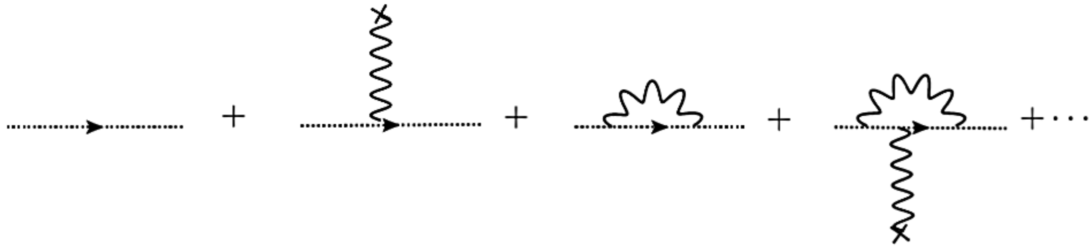


FIG. 1. Sum of Feynman diagrams for the dressed quenched propagator represented by a single path integral.

$$V^A_{\rm scal}[k,\varepsilon] \equiv \varepsilon_\mu \int_0^T d\tau\, \dot{x}^\mu(\tau) {\rm e}^{ik\cdot x(\tau)}. \tag{2.5}$$

According to our convention, external photon momenta are ingoing. Note that the integrand in S_i, as defined in Eq. (2.2), may also be written as

$$\frac{1}{4\pi^{\frac{D}{2}}}\left[\Gamma\left(\frac{D}{2}-1\right)\frac{\dot{x}_1\cdot\dot{x}_2}{[(x_1-x_2)^2]^{\frac{D}{2}-1}} - \frac{1-\xi}{4}\Gamma\left(\frac{D}{2}-2\right)\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}[(x_1-x_2)^2]^{2-\frac{D}{2}}\right], \tag{2.6}$$

which shows, already at this level, that a change of the covariant gauge parameter creates only a total derivative term.

The path integral is computed by splitting $x^\mu(\tau)$ into a "background" part $x^\mu_{\rm bg}(\tau)$, which encodes the boundary conditions, and a fluctuation part $q^\mu(\tau)$, which has Dirichlet boundary conditions at the end points $\tau = 0,\, T$,

$$\begin{aligned} x(\tau) &= x_{\rm bg}(\tau) + q(\tau),\\ x_{\rm bg}(\tau) &= x' + \frac{(x-x')\tau}{T},\\ \dot{x}(\tau) &= \frac{x-x'}{T} + \dot{q}(\tau),\\ q(0) &= q(T) = 0. \end{aligned} \tag{2.7}$$

The path integral over the fluctuation variable $q(\tau)$ is Gaussian, except for the denominators of the photon exchange terms S_i. A fully Gaussian representation is achieved by the further introduction of a photon proper time [35], rewriting

$$\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}([x(\tau_a)-x(\tau_b)]^2)^\lambda} = \int_0^\infty d\bar{T}(4\pi\bar{T})^{-\frac{D}{2}}\exp\left[-\frac{(x(\tau_a)-x(\tau_b))^2}{4\bar{T}}\right]. \tag{2.8}$$

The calculation of the path integral then requires only the knowledge of the free path integral normalization, which is

$$\int \mathcal{D}q(\tau){\rm e}^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2} = (4\pi T)^{-\frac{D}{2}}, \tag{2.9}$$

and of the two-point correlator, given by [48,49]

$$\langle q^\mu(\tau_1)q^\nu(\tau_2)\rangle = -2\delta^{\mu\nu}\Delta(\tau_1,\tau_2), \tag{2.10}$$

with the worldline Green function $\Delta(\tau_i,\tau_j)$,

$$\Delta(\tau_1,\tau_2) = \frac{\tau_1\tau_2}{T} + \frac{|\tau_1-\tau_2|}{2} - \frac{\tau_1+\tau_2}{2}. \tag{2.11}$$

We note that this Green function has a nontrivial coincidence limit

$$\Delta(\tau,\tau) = \frac{\tau^2}{T} - \tau, \tag{2.12}$$

and we will also need its following derivatives:

$$\begin{aligned} {}^\bullet\!\Delta(\tau_1,\tau_2) &= \frac{\tau_2}{T} + \frac{1}{2}{\rm sign}(\tau_1-\tau_2) - \frac{1}{2},\\ \Delta^\bullet(\tau_1,\tau_2) &= \frac{\tau_1}{T} - \frac{1}{2}{\rm sign}(\tau_1-\tau_2) - \frac{1}{2},\\ {}^\bullet\!\Delta^\bullet(\tau_1,\tau_2) &= \frac{1}{T} - \delta(\tau_1-\tau_2). \end{aligned} \tag{2.13}$$

Here we follow the notation [49] that left and right dots indicate derivatives with respect to the first argument and the second argument, respectively. Note that the mixed derivative ${}^\bullet\!\Delta^\bullet(\tau_1,\tau_2)$ contains a delta function which brings together two photon legs; this is how the seagull vertex arises in the worldline formalism.

In the simplest case, for the free scalar propagator, we thus get the following standard proper-time representation in D dimensions:

$$\Gamma_{\rm free}[x,x'] = \int_0^\infty dT\,{\rm e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}{\rm e}^{-\frac{1}{4T}(x-x')^2}. \tag{2.14}$$

III. LANDAU-KHALATNIKOV-FRADKIN TRANSFORMATIONS

LKFTs are rules which transform Green functions in a specific manner from one covariant gauge to another. The set of LKFT are nonperturbative in nature, and we have already discussed the history of their derivation and applications in Sec. I. It is clear from the pioneering works of Landau, Khalatnikov, and Fradkin [18,19] that these transformations work similarly for spinor and scalar QED. They give explicit rules with closed formulas to all orders in coordinate space for the two- and three-point functions. The momentum space treatment of the LKFT was carried out perturbatively at one- or two-loop orders in [21]. Their nonperturbative implementation in momentum space was performed numerically in QED_3 to establish the gauge invariance of chiral symmetry breaking and confinement [50,51].

Let us look at the derivation of the LKFT for the two-point propagator, following [18,19]. The photon propagator in the coordinate space can be written as

$$D_{\mu\nu}(x,f) = D_{\mu\nu}(x,0) + \partial_\mu \partial_\nu f_D(x), \tag{3.1}$$

where $f_D(x)$ is some function which corresponds to a particular gauge fixing procedure. In covariant gauge for D space-time dimension, its explicit form is

$$f_D(x) = -i\xi e^2 \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{\mathrm{e}^{-ik\cdot x}}{k^4}, \tag{3.2}$$

where μ is the usual mass scale, introduced to ensure that the coupling e remains dimensionless in every dimension D. By means of dimensional regularization, this integral can be evaluated to obtain

$$f_D(x) = -\frac{ie^2\xi}{16\pi^{\frac{D}{2}}}(\mu x)^{4-D}\Gamma\left(\frac{D}{2}-2\right). \tag{3.3}$$

In [7], LKFTs were used to obtain the covariant gauge representation of the fermion propagator, starting from its knowledge in the Landau gauge for $D = 3$ and $D = 4$. For the three-dimensional case, using $\alpha = e^2/(4\pi)$, one gets

$$f_3(x) = -\frac{i\alpha\xi x}{2}, \tag{3.4}$$

which leads to the following fermion propagator in an arbitrary gauge:

$$S_F(x;\xi) = S_F(x;0)\mathrm{e}^{-(\alpha\xi/2)x}. \tag{3.5}$$

Fourier transforming to momentum space, they recover the known results for the wave function renormalization $F(p;\xi)$ and the mass function $\mathcal{M}(p;\xi)$ at the one-loop order up to a term proportional to $\alpha\xi^0$ (see [7]) as permitted by the structure of the LKFT.

We now look at the four-dimensional case. By expanding Eq. (3.3) around $D = 4-\epsilon$, and using

$$\Gamma\left(-\frac{\epsilon}{2}\right) = -\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \tag{3.6}$$

we obtain

$$f_4(x) = i\frac{\xi e^2}{16\pi^2}\left[\frac{2}{\epsilon} + \gamma + \ln\pi + 2\ln(\mu x) + \mathcal{O}(\epsilon)\right]. \tag{3.7}$$

Thus in the four-dimensional case one has to be more careful. Note that in the term proportional to $\ln x$ one cannot simply put $x = 0$. Therefore, we need to introduce a cutoff scale x_{min} (see [7]). We then arrive at

$$f_4(x_{\mathrm{min}}) - f_4(x) = -i\ln\left(\frac{x^2}{x^2_{\mathrm{min}}}\right)^\kappa, \tag{3.8}$$

with $\kappa = \alpha\xi/(4\pi)$, and as long as we have the knowledge of the propagator in one gauge, we can transform it to any other gauge according to the formula

$$S_F(x;\xi) = S_F(x;0)\mathrm{e}^{-i(f_4(x_{\mathrm{min}})-f_4(x))} = S_F(x,0)\left(\frac{x^2}{x^2_{\mathrm{min}}}\right)^{-\kappa}. \tag{3.9}$$

And again, after Fourier transforming to momentum space, one obtains higher order information for $F(k;\xi)$ and $\mathcal{M}(k;\xi)$, whose lowest order values are $F(k,0) = 1$ and $\mathcal{M}(k;0) = 0$. It is shown in [7] that the perturbative expansion of the result in [7] for the fermion propagator matches onto its known one-loop results up to gauge independent terms at that order.

As is clear from Eq. (3.9), it encapsulates nonperturbative information about the propagator. One can Taylor expand $(\frac{x^2}{x^2_{\mathrm{min}}})^{-\kappa}$ and substitute it back into Eq. (3.9) to arrive at

$$S_F(x;\xi) = S_F(x;0)\left(1 - \log\left[\frac{x^2}{x^2_{\mathrm{min}}}\right]\kappa + \frac{1}{2}\log\left[\frac{x^2}{x^2_{\mathrm{min}}}\right]^2\kappa^2 + \cdots\right). \tag{3.10}$$

It has been argued in [21] that this expansion reproduces correct leading logarithms to any arbitrary order in perturbation theory. In [18,19], a formula for the behavior of the three-vertex under changes of the gauge parameter was also obtained, although only in coordinate space. However, its consequences for momentum space calculations have never been explored, and it is not straightforward to do so. This LKFT for the vertex is much easier to study in the worldline formalism, which is what we take up in Sec. VI. In order to do this, we start developing the formalism in the next section.

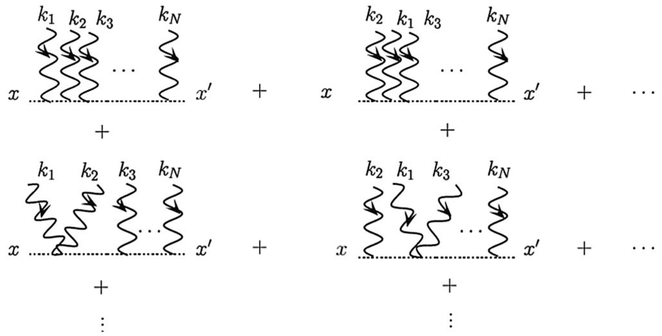


FIG. 2. Diagrams contributing to the N-propagator.

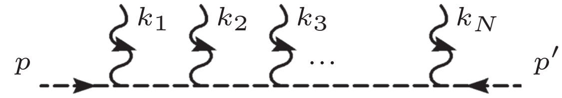


FIG. 3. Multiphoton Compton-scattering diagram.

IV. MULTIPHOTON AMPLITUDES IN SCALAR QED

We will now study the amplitude for the scalar to propagate in x-space from point x' to point x, and to absorb or emit N photons along the way with fixed momenta and polarizations (this object was called "N-propagator" in [37]). It corresponds to the diagrams shown in Fig. 2.

According to the diagrams in Fig. 2, the worldline representation of this amplitude is

$$\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT\,\mathrm{e}^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\mathrm{e}^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} \times \int_0^T \prod_{i=1}^N d\tau_i V_{\mathrm{scal}}^A[k_1,\varepsilon_1]\cdots V_{\mathrm{scal}}^A[k_N,\varepsilon_N], \tag{4.1}$$

where the photon vertex operator $V_{\mathrm{scal}}^A[k,\varepsilon]$ has been introduced in Eq. (2.5). We will now derive a closed-form expression for this amplitude. For this purpose, it will be convenient to formally rewrite the vertex operator as

$$V_{\mathrm{scal}}^A[k,\varepsilon] = \int_0^T d\tau \varepsilon\cdot\dot{x}(\tau)\mathrm{e}^{ik\cdot x(\tau)} = \int_0^T d\tau\,\mathrm{e}^{ik\cdot x(\tau)+\varepsilon\cdot\dot{x}(\tau)}\Big|_{\mathrm{lin}\,\varepsilon}. \tag{4.2}$$

Substituting this vertex operator in Eq. (4.1), and applying the split in Eq. (2.7), one gets

$$\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT\,\mathrm{e}^{-m^2T}\mathrm{e}^{-\frac{1}{4T}(x-x')^2} \int_{q(0)=q(T)=0} \mathcal{D}q(\tau)\mathrm{e}^{-\frac{1}{4}\int_0^T d\tau \dot{q}^2} \times \int_0^T \prod_{i=1}^N d\tau_i\,\mathrm{e}^{\Sigma_{i=1}^N(\varepsilon_i\cdot\frac{(x-x')}{T}+\varepsilon_i\cdot\dot{q}(\tau_i)+ik_i\cdot(x-x')\frac{\tau_i}{T}+ik_i\cdot x'+ik_i\cdot q(\tau_i))}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}. \tag{4.3}$$

After completing the square in the exponential, we obtain the following tree-level "Bern-Kosower-type formula" in configuration space:

$$\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT\,\mathrm{e}^{-m^2T}\mathrm{e}^{-\frac{1}{4T}(x-x')^2}(4\pi T)^{-\frac{D}{2}} \times \int_0^T \prod_{i=1}^N d\tau_i \mathrm{e}^{\Sigma_{i=1}^N(\varepsilon_i\cdot\frac{(x-x')}{T}+ik_i\cdot(x-x')\frac{\tau_i}{T}+ik_i\cdot x')}\mathrm{e}^{\Sigma_{i,j=1}^N[\Delta_{ij}k_i\cdot k_j-2i\,{}^\bullet\!\Delta_{ij}\varepsilon_i\cdot k_j-{}^\bullet\!\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}. \tag{4.4}$$

Now, we also Fourier transform the scalar legs of the master formula in Eq. (4.4) to momentum space,

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = \int d^Dx \int d^Dx'\mathrm{e}^{ip\cdot x+ip'\cdot x'}\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N]. \tag{4.5}$$

This gives a representation of the multiphoton Compton scattering diagram as depicted in Fig. 3 (together with all the permuted and "seagulled" ones).

Changing the integration variables x, x' to

$$x - x' = x_- \quad \text{and} \quad x + x' = 2x_+,$$

the integral over x_+ just produces the usual energy-momentum conservation factor,

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta^D\left(p+p'+\sum_{i=1}^N k_i\right)\int_0^\infty dT\,\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\int d^Dx_-\mathrm{e}^{-\frac{1}{4T}x_-^2} \times \int_0^T \prod_{i=1}^N d\tau_i\,\mathrm{e}^{ix_-\cdot(p+\Sigma_{i=1}^N\frac{k_i\tau_i}{T})}\mathrm{e}^{\Sigma_{i=1}^N\frac{\varepsilon_i\cdot x_-}{T}}\mathrm{e}^{\Sigma_{i,j=1}^N[\Delta_{ij}k_i\cdot k_j-2i\,{}^\bullet\!\Delta_{ij}\varepsilon_i\cdot k_j-{}^\bullet\!\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}. \tag{4.6}$$

After also performing the x_- integral, and some rearrangements, one arrives at

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta^D\left(p+p'+\sum_{i=1}^N k_i\right)\int_0^\infty dT\,\mathrm{e}^{-T(m^2+p^2)}$$
$$\times\int_0^T\prod_{i=1}^N d\tau_i \mathrm{e}^{\Sigma_{i=1}^N(-2k_i\cdot p\tau_i+2i\varepsilon_i\cdot p)+\Sigma_{i,j=1}^N[(\frac{|\tau_i-\tau_j|}{2}-\frac{\tau_i+\tau_j}{2})k_i\cdot k_j-i(\mathrm{sign}(\tau_i-\tau_j)-1)\varepsilon_i\cdot k_j+\delta(\tau_i-\tau_j)\varepsilon_i\cdot\varepsilon_j]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}. \tag{4.7}$$

This is our final representation of the N-propagator in momentum space. It is important to mention that it gives the *untruncated* propagator, including the final scalar propagators on both ends. On shell it corresponds to multiphoton Compton scattering, while off shell it can be used for constructing higher-loop amplitudes by sewing. Since this momentum space version involves the integration variables only linearly in the exponent, for any given ordering of the photon legs it is straightforward to do the integrals and verify that they correspond to the usual sum of Feynman diagrams. The main point of formula (4.7) is its ability to combine all the $N!$ orderings. This may not appear very relevant at tree level, but when used as a building block for higher-loop amplitudes, it leads to integral representations for nontrivial sums of diagrams. For example, taking two copies of the N-propagator, pairing off the photons on each side, and connecting them by free photon propagators, we can construct an integral representation of the sum of ladder plus crossed-ladder diagrams, important for the study of scalar bound states in scalar QED. For the case of scalar field theory, the usefulness of this construction has been demonstrated in [37,52].

Let us also remark that the master formula (4.7) can be written even more compactly at the expense of introducing some more notation. Namely, defining

$$\begin{aligned} K_0 &\equiv p, \\ K_i &\equiv k_i, \quad i=1,\ldots,N, \\ K_{N+1} &\equiv p', \end{aligned} \tag{4.8}$$

as well as $\tau_0 = T$, $\tau_{N+1} = 0$ and $\varepsilon_0 = \varepsilon_{N+1} = 0$ the exponent of the master formula can be rewritten with the help of energy-momentum conservation, such as to arrive at the following form:

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta^D\left(p+p'+\sum_{i=1}^N k_i\right)\int_0^\infty dT\,\mathrm{e}^{-m^2T}$$
$$\times\int_0^T\prod_{i=1}^N d\tau_i\,\mathrm{e}^{\Sigma_{i,j=0}^{N+1}[\frac{1}{2}|\tau_i-\tau_j|K_i\cdot K_j-i\mathrm{sign}(\tau_i-\tau_j)\varepsilon_i\cdot K_j+\delta(\tau_i-\tau_j)\varepsilon_i\cdot\varepsilon_j]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}. \tag{4.9}$$

This form of the momentum space master formula has been previously obtained by Daikouji *et al.* [36] by a direct comparison with the corresponding Feynman-Schwinger parameter integrals.

V. THE ONE-LOOP PROPAGATOR AND VERTEX IN FEYNMAN GAUGE

We will now apply the master formula (4.7) to a rederivation of the one-loop scalar propagator and vertex, at first in Feynman gauge. Usually such worldline master formulas are used for a direct calculation in parameter space, and the fact that part or all of the momentum integrals of the corresponding Feynman diagrams have effectively already been done is an advantage. Here we will, instead, be satisfied with showing that the master formula correctly reproduces those Feynman integrals. This not only will provide us with a check but also will allow us to draw on the results of [12].

The one-loop scalar propagator, shown in Fig. 4 (there is also a second diagram with a seagull vertex, which, however, vanishes in dimensional regularization), could be obtained either from Eq. (2.1), with a single factor of S_i, and Fourier transformation, or from the master formula (4.7) with $N=2$ by sewing off the two photons legs. We prefer to take the second route here. Let us do this first in Feynman gauge.

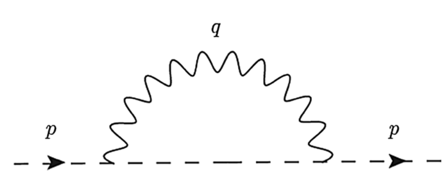


FIG. 4. One-loop correction to the scalar propagator.

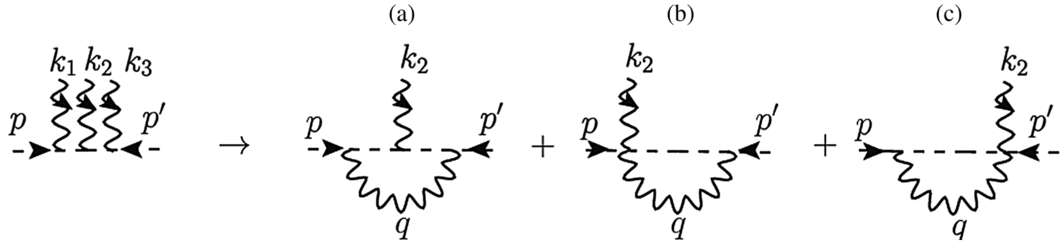


FIG. 5. One-loop 1PI vertex from the $N=3$ master formula.

The sewing is done through setting

$$\varepsilon_1^\mu \varepsilon_2^\nu \to \frac{\delta_{\mu\nu}}{q^2}, \tag{5.1}$$

where $k_1 = q$, $k_2 = -q$, and the photon momentum q is integrated over. After the rescaling $\tau_i = Tu_i$, this yields

$$\Gamma^{\text{no-trunc}}_{\text{Feyn}}(p) = e^2 \int_0^\infty dT T^2 \mathrm{e}^{-T(m^2+p^2)} \int_0^1 du_1 \int_0^{u_1} du_2 \\ \times \int \frac{d^D q}{(2\pi)^D} \left[(2p+q)^\mu (2p+q)^\nu - \frac{2}{T}\delta(u_1-u_2)\delta_{\mu\nu} \right] \frac{\delta^{\mu\nu}}{q^2} \mathrm{e}^{-T(u_1-u_2)(q^2+2p\cdot q)}. \tag{5.2}$$

The superscript "no-trunc" refers to the above-mentioned fact that this expression still includes the two external propagators. The term with the delta function corresponds to the seagull diagram and can be omitted. The parameter integrals give

$$\int_0^\infty dT T^2 \mathrm{e}^{-T(m^2+p^2)} \int_0^1 du_1 \int_0^{u_1} du_2 \mathrm{e}^{-T(u_1-u_2)(q^2+2p\cdot q)} = \frac{1}{(m^2+p^2)^2[(p+q)^2+m^2]}. \tag{5.3}$$

Thus we get

$$\Gamma^{\text{no-trunc}}_{\text{Feyn}}(p) = e^2 \frac{1}{(p^2+m^2)^2} \int \frac{d^D q}{(2\pi)^D} \frac{(2p+q)^2}{q^2[(p+q)^2+m^2]}. \tag{5.4}$$

The integral can be done using the list of integrals given in Appendix A, leading to

$$\Gamma^{\text{no-trunc}}_{\text{Feyn}}(p) = \frac{1}{(p^2+m^2)^2} \frac{e^2}{(4\pi)^{\frac{D}{2}}} (m^2)^{\frac{D}{2}-1} \Gamma\left(1-\frac{D}{2}\right) \left[2\frac{(m^2-p^2)}{m^2} {}_2F_1\left(2-\frac{D}{2}, 1; \frac{D}{2}; -\frac{p^2}{m^2}\right) - 1 \right]. \tag{5.5}$$

This agrees with the result in [12] after continuation to Minkowski space and removal of the external propagators.

Now let us look at the scalar-photon vertex. It can be obtained from Eq. (4.7) with $N=3$ and the standard ordering $(\tau_1 \geq \tau_2 \geq \tau_3)$ by sewing photon 1 and photon 3. Our interest is in the form factor decomposition of the 1PI vertex. Diagrammatically, the one-particle irreducible (1PI) vertex is given by the three diagrams a, b, and c depicted in Fig. 5. Our sewing procedure (still using Feynman gauge) generates these diagrams in the form

$$\Gamma_{\text{vertex}}[p, p'; k_2, \varepsilon_2] = \Gamma_a[p, p'; k_2, \varepsilon_2] + \Gamma_b[p, p'; k_2, \varepsilon_2] + \Gamma_c[p, p'; k_2, \varepsilon_2] \\ = -e^3(m^2+p^2)(m^2+p'^2) \int_0^\infty dT\, \mathrm{e}^{-T(m^2+p^2)} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \\ \times \int \frac{d^D q}{(2\pi)^D} \left\{ \frac{(l_1\cdot l_3)(l_2\cdot\varepsilon_2)}{q^2} - \frac{(l_3\cdot\varepsilon_2)}{q^2} 2\delta(\tau_1-\tau_2) + \frac{(l_1\cdot\varepsilon_2)}{q^2} 2\delta(\tau_2-\tau_3) \right\} \\ \times \mathrm{e}^{-(-2q\cdot p+q^2)\tau_1 - (2k_2\cdot p + k_2^2 - 2q\cdot k_2)\tau_2 - (-q^2+2q\cdot(p+k_2))\tau_3}, \tag{5.6}$$

where $(q = -k_1 = k_3)$

$$
\begin{aligned}
l_1 &= 2p - q, \\
l_2 &= k_2 + 2(p-q), \\
l_3 &= 2p' + q.
\end{aligned}
\tag{5.7}
$$

The first term inside the curly bracket represents the a diagram, the second term which contains $\delta(\tau_1 - \tau_2)$ corresponds to the b diagram and the last term corresponds to the c diagram. The external propagators have already been removed multiplying by a factor of $-(m^2+p^2)(m^2+p'^2)$. Let us first look at diagram a,

$$
\begin{aligned}
\Gamma_a[p,p';k_2,\varepsilon_2] = {}& -e^3(m^2+p^2)(m^2+p'^2)\int_0^\infty dT T^3 \mathrm{e}^{-T(m^2+p^2)}\int_0^1 du_1\int_0^{u_1} du_2 \int_0^{u_2} du_3 \\
&\times \int \frac{d^Dq}{(2\pi)^D}\frac{(l_1\cdot l_3)(l_2\cdot\varepsilon_2)}{q^2}\mathrm{e}^{-(-2q\cdot p+q^2)Tu_1-(2k_2\cdot p+k_2^2-2q\cdot k_2)Tu_2-(-q^2+2q\cdot(p+k_2))Tu_3},
\end{aligned}
\tag{5.8}
$$

where

$$
\begin{aligned}
l_1\cdot l_3 &= -q^2 + 2q\cdot(p-p') + 4p'\cdot p, \\
l_2\cdot\varepsilon_2 &= k_2\cdot\varepsilon_2 + 2(p-q)\cdot\varepsilon_2.
\end{aligned}
\tag{5.9}
$$

Performing the T and u_i-integrals leads to

$$
\Gamma_a^\mu[p,p';k_2] \overset{u_1>u_2>u_3}{=} -e^3\int\frac{d^Dq}{(2\pi)^D}\frac{l_1\cdot l_3}{q^2}l_2^\mu\frac{1}{(m^2+p^2-2q\cdot p+q^2)\underbrace{[m^2+p^2+q^2+2k_2\cdot p+k_2^2-2q\cdot(k_2+p)]}_{m^2+(k_2+p-q)^2=m^2+(p'+q)^2}},
\tag{5.10}
$$

($\Gamma_a \equiv \varepsilon_{2\mu}\Gamma_a^\mu$). Note that the q-integral can be rewritten as

$$
\begin{aligned}
\int d^Dq\frac{(l_1\cdot l_3)l_2^\mu}{q^2[m^2+(p-q)^2][m^2+(p'+q)^2]} &= \int d^Dq\left[\frac{-q^2+2q\cdot(p-p')+4p\cdot p'}{q^2[m^2+(p-q)^2][m^2+(p'+q)^2]}\right](k_2^\mu - 2q^\mu + 2p^\mu) \\
&= -(k_2^\mu+2p^\mu)K^{(0)} + 2K_\mu^{(1)} + 2(p^\nu - p'^\nu)[(k_2^\mu+2p^\mu)J_\nu^{(1)} - 2J_{\mu\nu}^{(2)}] \\
&\quad + 4p\cdot p'[(k_2^\mu+2p^\mu)J^{(0)} - 2J_\mu^{(1)}],
\end{aligned}
\tag{5.11}
$$

where

$$
\begin{aligned}
K^{(0)} &= \int d^Dq\frac{1}{[m^2+(p-q)^2][m^2+(q+p')^2]}, \\
K_\mu^{(1)} &= \int d^Dq\frac{q_\mu}{[m^2+(p-q)^2][m^2+(q+p')^2]}, \\
J^{(0)} &= \int d^Dq\frac{1}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]}, \\
J_\mu^{(1)} &= \int d^Dq\frac{q_\mu}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]}, \\
J_{\mu\nu}^{(2)} &= \int d^Dq\frac{q_\mu q_\nu}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]}.
\end{aligned}
\tag{5.12}
$$

The final result for diagram a becomes

$$
\begin{aligned}
\Gamma_a^\mu[p,p';k_2] = -\frac{e^3}{(2\pi)^D}\{&(p'^\mu - p^\mu)K^{(0)} + 2K_\mu^{(1)} \\
&+ 2(p^\nu - p'^\nu)[(p^\mu - p'^\mu)J_\nu^{(1)} - 2J_{\mu\nu}^{(2)}] \\
&+ 4(p\cdot p')[(p^\mu - p'^\mu)J^{(0)} - 2J_\mu^{(1)}]\}.
\end{aligned}
\tag{5.13}
$$

The parameter integrals which appear for the calculation of diagram b are similar to the ones for the scalar propagator. Let us present here just the final result, relegating the details to Appendix B,

$$
\begin{aligned}
\Gamma_b^\mu(p') = \frac{1}{2}\frac{e^3 m^{D-4}p'^\mu}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\Bigg\{&\left(\frac{m^2}{p'^2}-3\right) \\
&\times {}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p'^2}{m^2}\right) - \frac{m^2}{p'^2}\Bigg\}.
\end{aligned}
\tag{5.14}
$$

Diagram c is obtained from diagram b simply by the replacement

$$\Gamma_c^\mu = \Gamma_b^\mu(p' \to -p). \tag{5.15}$$

All these results are in agreement with the ones presented in [12].

VI. GENERALIZATION OF THE LANDAU-KHALATNIKOV-FRADKIN TRANSFORMATION

We now come to our central topic, which is the generalization of the LKFT to arbitrary amplitudes in scalar QED [we consider scalar QED as a pure gauge theory only, without the induced $\lambda(\phi^*\phi)^2$ coupling term].

As we have seen, each external photon will be represented by a vertex operator, Eq. (2.5), inserted either on a scalar loop or on a line. A gauge transformation $\varepsilon_i \to \varepsilon_i + \xi k_i$ applied to the ith photon creates the integral of a total derivative that collapses to boundary terms,

$$\begin{aligned} V_{\rm scal}[\varepsilon_i, k_i] &= \int_0^T d\tau_i \varepsilon_{i\mu}\dot{x}_i^\mu \mathrm{e}^{ik_i\cdot x(\tau_i)} \to V_{\rm scal}[\varepsilon_i, k_i] \\ &\quad - i\xi \int_0^T \frac{\partial}{\partial\tau_i}\mathrm{e}^{ik_i\cdot x(\tau_i)} \\ &= V_{\rm scal}[\varepsilon_i, k_i] - i\xi(\mathrm{e}^{ik_i\cdot x} - \mathrm{e}^{ik_i\cdot x'}). \end{aligned} \tag{6.1}$$

Thus, for the closed loop, these boundary terms cancel. For the open line case they remain but do not contribute to on-shell matrix elements. This is, of course, just the QED Ward identity, which we need not consider here further.

More interesting is the case of a change of gauge for all the internal photons. Each internal photon is represented by a factor of $-S_i$ with S_i as given in Eq. (2.2). And from Eq. (2.6), we can see that a change in the gauge parameter ξ by $\Delta\xi$ will change S_i by

$$\begin{aligned} \Delta_\xi S_i &= \Delta\xi \frac{e^2}{32\pi^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-2\right)\int_0^T d\tau_1 \\ &\quad \times \int_0^T d\tau_2 \frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}[(x_1-x_2)^2]^{2-\frac{D}{2}}. \end{aligned} \tag{6.2}$$

Since the integrand is a total derivative in both variables, if the photon at least on one end sits on a closed loop, such as in Fig. 6, the result will vanish. Therefore, the gauge transformation properties of an amplitude are determined by the photons exchanged between two scalar lines, or along one scalar line. Thus, in the study of the gauge parameter dependence, we can disregard external photons as well as closed scalar loops, and it therefore suffices to study the quenched $2n$ scalar amplitude. This amplitude, which we will denote by $A^{\rm qu}(x_1, \ldots, x_n; x_1', \ldots, x_n'|\xi)$, has $n!$ contributions, corresponding to the ways the fields $\phi(x_1), \ldots, \phi(x_n)$ can be matched with the conjugate complex fields $\phi^*(x_1'), \ldots, \phi^*(x_n')$,

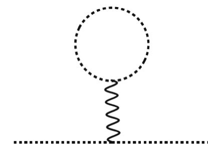


FIG. 6. An internal photon with one end on a scalar line and the other one on a loop.

$$\begin{aligned} &A^{\rm qu}(x_1, \ldots, x_n; x_1', \ldots, x_n'|\xi) \\ &\quad = \sum_{\pi\in S_n} A_\pi^{\rm qu}(x_1, \ldots, x_n; x_{\pi(1)}', \ldots, x_{\pi(n)}'|\xi), \end{aligned} \tag{6.3}$$

where in the partial amplitude $A_\pi^{\rm qu}(x_1, \ldots, x_n; x_{\pi(1)}', \ldots, x_{\pi(n)}'|\xi)$ it is understood that the line ending at x_i starts at $x_{\pi(i)}'$. The worldline representation of this partial amplitude at the quenched level is [14] [compare Eq. (2.2)]

$$\begin{aligned} &A_\pi^{\rm qu}(x_1, \ldots, x_n; x_{\pi(1)}', \ldots, x_{\pi(n)}'|\xi) \\ &\quad = \prod_{l=1}^n \int_0^\infty dT_l \mathrm{e}^{-m^2T_l} \\ &\qquad \times \int_{x_l(0)=x_{\pi(l)}'}^{x_l(T_l)=x_l} \mathcal{D}x_l(\tau_l)\mathrm{e}^{-\Sigma_{l=1}^n S_0^{(l)} - \Sigma_{k,l=1}^n S_{i\pi}^{(k,l)}}. \end{aligned} \tag{6.4}$$

Here, $S_0^{(l)}$ is the free worldline Lagrangian for the path integral representing line l,

$$S_0^{(l)} = \int_0^{T_l} d\tau_l \frac{1}{4}\dot{x}_l^2, \tag{6.5}$$

and $S_{i\pi}^{(k,l)}$ generates all the photons connecting lines k and l,

$$S_{i\pi}^{(k,l)} = \frac{e^2}{2}\int_0^{T_k} d\tau_k \int_0^{T_l} d\tau_l \dot{x}_k^\mu D_{\mu\nu}(x_k - x_l)\dot{x}_l^\nu. \tag{6.6}$$

Thus, after a gauge change,

$$\begin{aligned} &A_\pi^{\rm qu}(x_1, \ldots, x_n; x_{\pi(1)}', \ldots, x_{\pi(n)}'|\xi + \Delta\xi) \\ &\quad = \prod_{l=1}^n \int_0^\infty dT_l \mathrm{e}^{-m^2T_l} \\ &\qquad \times \int_{x_l(0)=x_{\pi(l)}'}^{x_l(T_l)=x_l} \mathcal{D}x_l(\tau_l)\mathrm{e}^{-\Sigma_{l=1}^n S_0^{(l)} - \Sigma_{k,l=1}^n (S_{i\pi}^{(k,l)} + \Delta_\xi S_{i\pi}^{(k,l)})}, \end{aligned} \tag{6.7}$$

where, from Eq. (6.2), we have

$$
\begin{aligned}
\Delta_\xi S^{(k,l)}_{i\pi} &= \Delta\xi \frac{e^2}{32\pi^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-2\right)\{[(x_k(T_k)-x_l(T_l))^2]^{2-D/2} \\
&\quad -[(x_k(T_k)-x_l(0))^2]^{2-D/2}-[(x_k(0)-x_l(T_l))^2]^{2-D/2} \\
&\quad +[(x_k(0)-x_l(0))^2]^{2-D/2}\} \\
&= \Delta\xi \frac{e^2}{32\pi^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-2\right)\{[(x_k-x_l)^2]^{2-D/2} \\
&\quad -[(x_k-x'_{\pi(l)})^2]^{2-D/2}-[(x'_{\pi(k)}-x_l)^2]^{2-D/2} \\
&\quad +[(x'_{\pi(k)}-x'_{\pi(l)})^2]^{2-D/2}\}. \qquad (6.8)
\end{aligned}
$$

Since this depends only on the end points of the scalar trajectories, we can pull the factors involving $\Delta\xi$ in Eq. (6.7) out of the path integration, leading to

$$
\begin{aligned}
&A^{\rm qu}_\pi(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi+\Delta\xi) \\
&\quad = T_\pi A^{\rm qu}_\pi(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi), \qquad (6.9)
\end{aligned}
$$

where

$$T_\pi \equiv \prod_{k,l=1}^{N} {\rm e}^{-\Delta_\xi S^{(k,l)}_{i\pi}}. \qquad (6.10)$$

This is an exact D-dimensional result. When using it in dimensional regularization around $D=4$, one has to take into account that the full nonperturbative $A^{\rm qu}$ in scalar QED has poles in ϵ to arbitrary order, so that also the prefactor T_π, although regular, needs to be kept to all orders. Here we will consider only the leading constant term of this prefactor. Thus, we compute

$$\lim_{D\to4}{\rm e}^{-\Delta_\xi S^{(k,l)}_{i\pi}} = (r^{(k,l)}_\pi)^c, \qquad (6.11)$$

where we have introduced the constant

$$c \equiv \Delta\xi \frac{e^2}{32\pi^2}, \qquad (6.12)$$

and the conformal cross ratio $r^{(k,l)}_\pi$ associated with the four end points of the lines k and l,

$$r^{(k,l)}_\pi \equiv \frac{(x_k-x_l)^2(x'_{\pi(k)}-x'_{\pi(l)})^2}{(x'_{\pi(k)}-x_l)^2(x_k-x'_{\pi(l)})^2}. \qquad (6.13)$$

Thus, at the leading order, the prefactor turns into

$$T_\pi = \left(\prod_{k,l=1}^{N} r^{(k,l)}_\pi\right)^c + O(\epsilon). \qquad (6.14)$$

We note that for the case of a single propagator, $s=k=l=1$, Eq. (6.14) degenerates into

$$T = \left[\frac{(x-x)^2(x'-x')^2}{((x-x')^2)^2}\right]^c. \qquad (6.15)$$

Therefore, if we replace the vanishing numerator $(x-x)^2(x'-x')^2$ by the cutoff $(x^2_{\rm min})^2$, and $\Delta\xi$ by ξ, we recuperate the original LKFT, Eq. (3.9).

VII. THE GENERALIZED LKFT IN PERTURBATION THEORY

As in the case of the original LKFT, one would like to know how the nonperturbative gauge transformation formula in Eq. (6.9) works out in perturbation theory. From the structure of that formula, it is immediately obvious that, given a calculation of a full x-space amplitude in scalar QED at a given loop level, it allows one to predict certain higher-order terms, albeit only gauge-dependent ones. More relevant from a practical point of view is, however, the change of gauge parameter at a fixed loop level. How this works diagrammatically should be clear from the above, but let us illustrate it with an example. Consider the 12-loop contribution to the scalar six-point function shown in Fig. 7, where it should be understood that we consider the sum of this diagram together with all the ones that differ from it only by "letting photon legs slide along scalar lines."

A gauge parameter change will affect all the photons, except the ones ending on a loop, and convert a photon connecting lines k and l into a factor $-\Delta_\xi S^{(k,l)}_{i\pi}$. Thus, the difference between gauges involves only lower-loop diagrams such as shown in Fig. 8. If the gauge transformation of the whole set of diagrams is called Δ_ξFig 7, we can write

$$
\begin{aligned}
\Delta_\xi{\rm Fig\,7} &= (-2\Delta_\xi S^{(1,2)}_{i\pi}){\rm Fig\,8}(a) + (-\Delta_\xi S^{(1,1)}_{i\pi}){\rm Fig\,8}(b) \\
&\quad + (-2\Delta_\xi S^{(1,3)}_{i\pi}){\rm Fig\,8}(c) + \cdots \\
&\quad + (-2\Delta_\xi S^{(1,2)}_{i\pi})(-\Delta_\xi S^{(1,1)}_{i\pi}){\rm Fig\,8}(d) + \cdots \\
&\quad + (-2\Delta_\xi S^{(1,2)}_{i\pi})(-\Delta_\xi S^{(1,1)}_{i\pi}) \\
&\quad \times (-2\Delta_\xi S^{(1,3)}_{i\pi}){\rm Fig\,8}(e) + \cdots \\
&\quad + \cdots. \qquad (7.1)
\end{aligned}
$$

Here, on the right-hand side the first line is for gauge transformation of each photon, one by one. In the second line, we have simultaneous gauge transformations of all possible pairs of photons, etc.

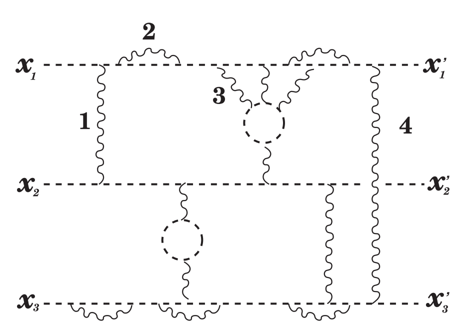
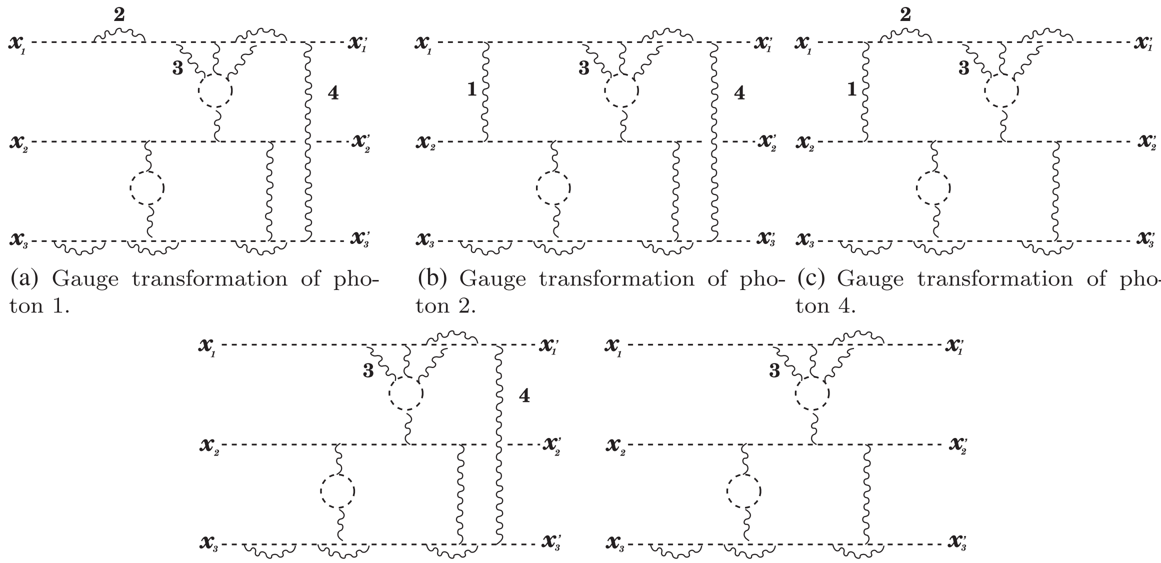


FIG. 7. Feynman diagram representing a class of contributions to the six-scalar amplitude at 12 loops.



(a) Gauge transformation of photon 1.

(b) Gauge transformation of photon 2.

(c) Gauge transformation of photon 4.

(d) Simultaneous gauge transformation of photons 1 and 2.

(e) Simultaneous gauge transformation of photons 1, 2 and 4.

FIG. 8. Gauge transformation of internal photons.

VIII. THE ONE-LOOP PROPAGATOR AND VERTEX IN ANY COVARIANT GAUGE

As has already been emphasized, mathematically the LKFT is more difficult to implement in momentum than in configuration space, and this remains true in our present more general framework. The nonperturbative transformation formula (6.9) is not amenable to a direct Fourier transformation, and neither does there appear to be an analogue of the perturbative formula (7.1). Nevertheless, the basic fact that gauge parameter transformations generate only total derivative terms in the worldline integrals applies, of course, also in momentum space. This deserves further investigation; here, we will be satisfied with identifying those total derivative terms for the case of the one-loop propagator and vertex, and using them for shifting the above results for Feynman gauge to an arbitrary covariant gauge in an efficient way.

For this purpose, let us go back to the scalar propagator, Fig. 4. We now use an arbitrary covariant gauge to sew the two photons together,

$$\varepsilon_1^\mu \varepsilon_2^\nu \to \frac{\delta^{\mu\nu} q^2 - (1-\xi) q^\mu q^\nu}{q^4}. \tag{8.1}$$

If we repeat our calculations of Sec. V in an arbitrary covariant gauge, we get

$$\Gamma_{\rm propagator}(p) = \Gamma_{\rm Feynman} + \Gamma_\xi = -e^2 (m^2+p^2)^2 \int_0^\infty dT T^2 \mathrm{e}^{-T(m^2+p^2)} \int_0^1 du_1 \int_0^{u_1} du_2 \\ \times \int \frac{d^D q}{(2\pi)^D} (2p^\mu + q^\mu)(2p^\nu + q^\nu) \left[\frac{\delta^{\mu\nu}}{q^2} + (\xi - 1) \frac{q^\mu q^\nu}{q^4} \right] \mathrm{e}^{-T(u_1-u_2)(q^2+2p\cdot q)}, \tag{8.2}$$

where $\Gamma_{\rm Feynman}$ is the part which contains $\delta^{\mu\nu}$ and Γ_ξ is the gauge part. Note that the gauge part can be written as a second derivative of the exponential as

$$(2p^\mu + q^\mu)(2p^\nu + q^\nu)(\xi - 1)\frac{q^\mu q^\nu}{q^4} \mathrm{e}^{-T(u_1-u_2)(q^2+2p\cdot q)} = (\xi - 1)\left[\frac{2p\cdot q}{q^2} + 1\right]^2 \mathrm{e}^{-T(u_1-u_2)(q^2+2p\cdot q)} \\ = -\frac{(\xi-1)}{T^2 q^4} \frac{\partial^2}{\partial u_1 \partial u_2} \mathrm{e}^{-T(u_1-u_2)(q^2+2p\cdot q)}. \tag{8.3}$$

Integrating this over u_1 and u_2 yields

$$\int_0^1 du_1 \int_0^{u_1} du_2 \frac{\partial^2}{\partial u_1 \partial u_2} \mathrm{e}^{-T(u_1-u_2)(q^2+2p\cdot q)} = 1 - T(q^2 + 2p\cdot q) - \mathrm{e}^{-T(q^2+2p\cdot q)}. \tag{8.4}$$

The first two terms lead to q-integrals that vanish in dimensional regularization. The third one gives a standard scalar two-point integral that is easy to compute, leading to our following final result for the propagator:

$$\Gamma_{\text{propagator}}(p) = \Gamma_{\text{Feynman}} + \Gamma_{\xi} = \frac{e^2}{m^2}\left(\frac{m^2}{4\pi}\right)^{\frac{D}{2}}\Gamma\left(1-\frac{D}{2}\right)\left\{1 - 2\frac{(m^2-p^2)}{m^2}\,{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right) + (1-\xi)\frac{(m^2+p^2)^2}{m^4}\,{}_2F_1\left(3-\frac{D}{2},2;\frac{D}{2};-\frac{p^2}{m^2}\right)\right\}. \tag{8.5}$$

Now, let us look at the scalar-photon vertex, Fig. 5. In an arbitrary covariant gauge, (5.6) becomes

$$\begin{aligned}\Gamma_{\text{vertex}}[p,p';k_2,\varepsilon_2] &= \Gamma_a[p,p';k_2,\varepsilon_2] + \Gamma_b[p,p';k_2,\varepsilon_2] + \Gamma_c[p,p';k_2,\varepsilon_2]\\ &= -e^3(m^2+p'^2)(m^2+p^2)\int_0^\infty dT\,\mathrm{e}^{-T(m^2+p^2)}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\int_0^{\tau_2} d\tau_3\\ &\quad\times\int\frac{d^Dq}{(2\pi)^D}\left\{\left[\frac{(l_1\cdot l_3)}{q^2} - (1-\xi)\frac{(l_1\cdot q)(l_3\cdot q)}{q^4}\right](l_2\cdot\varepsilon_2) - 2\delta(\tau_1-\tau_2)\left[\frac{l_3\cdot\varepsilon_2}{q^2} - (1-\xi)\frac{(l_3\cdot q)(\varepsilon_2\cdot q)}{q^4}\right]\right.\\ &\quad\left. + 2\delta(\tau_2-\tau_3)\left[\frac{(l_1\cdot\varepsilon_2)}{q^2} - (1-\xi)\frac{(l_1\cdot q)(\varepsilon_2\cdot q)}{q^4}\right]\right\}\mathrm{e}^{-(-2q\cdot p+q^2)\tau_1-(2k_2\cdot p+k_2^2-2q\cdot k_2)\tau_2-(-q^2+2q\cdot(p+k_2))\tau_3}.\end{aligned} \tag{8.6}$$

In the same way as for the propagator above, the gauge parameter dependent part can be obtained from two total derivatives of the exponential as

$$\begin{aligned}&(1-\xi)\frac{l_1\cdot q l_3\cdot q}{q^4}[\mathrm{e}^{(2q\cdot p-q^2)Tu_1-(2k_2\cdot p+k_2^2-2q\cdot k_2)Tu_2+(q^2+2q\cdot p')Tu_3}]\\ &\quad= (1-\xi)\frac{1}{T^2q^4}\left(\frac{\partial^2}{\partial u_1\partial u_3}\right)[\mathrm{e}^{(2q\cdot p-q^2)Tu_1-(2k_2\cdot p+k_2^2-2q\cdot k_2)Tu_2+(q^2+2q\cdot p')Tu_3}].\end{aligned} \tag{8.7}$$

Finally, the diagram a in a covariant gauge can be written as (see Appendix B for the details)

$$\begin{aligned}\Gamma_a^\mu[p,p';k_2] &= -\frac{e^3}{(2\pi)^D}\left\{(p'^\mu-p^\mu)K^{(0)} + 2K_\mu^{(1)} + 2(p^\nu-p'^\nu)[(p^\mu-p'^\mu)J_\nu^{(1)} - 2J_{\mu\nu}^{(2)}] + 4p\cdot p'[(p^\mu-p'^\mu)J^{(0)} - 2J_\mu^{(1)}]\right.\\ &\quad - (\xi-1)(p'^2+m^2)(p^2+m^2)\left\{\left[\frac{\pi^{\frac{D}{2}}(p'^\mu p^2+p^\mu m^2)}{p^2(p'^2+m^2)}\Gamma\left(1-\frac{D}{2}\right)(m^2)^{\frac{D}{2}-3}\,{}_2F_1\left(3-\frac{D}{2},2;\frac{D}{2};-\frac{p^2}{m^2}\right) - (p\leftrightarrow p')\right]\right.\\ &\quad\left.\left. - \left[\frac{\pi^{\frac{D}{2}}p^\mu}{p^2(p'^2+m^2)}\Gamma\left(1-\frac{D}{2}\right)(m^2)^{\frac{D}{2}-2}\,{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right) - (p\leftrightarrow p')\right] + (p^\mu-p'^\mu)I^{(0)} - 2I_\mu^{(1)}\right\}\right\},\end{aligned} \tag{8.8}$$

where, besides the integrals that already appeared in (5.12), we need to evaluate the following ones:

$$\begin{aligned}I^{(0)} &= \int d^Dq\,\frac{1}{q^4[m^2+(p-q)^2][m^2+(q+p')^2]},\\ I_\mu^{(1)} &= \int d^Dq\,\frac{q_\mu}{q^4[m^2+(p-q)^2][m^2+(q+p')^2]}.\end{aligned} \tag{8.9}$$

Similarly, diagram b yields (see Appendix B)

$$\begin{aligned}\Gamma_b^\mu(p') &= \frac{1}{2}\frac{e^3m^{D-4}\Gamma(1-\frac{D}{2})p'^\mu}{(4\pi)^{\frac{D}{2}}}\left\{\left(\frac{m^2}{p'^2}-3\right){}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p'^2}{m^2}\right) - \frac{m^2}{p'^2}\right.\\ &\quad\left. - (\xi-1)\left(\frac{p'^2+m^2}{p'^2}\right)\left[{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p'^2}{m^2}\right) - \left(\frac{p'^2+m^2}{m^2}\right){}_2F_1\left(3-\frac{D}{2},2;\frac{D}{2};-\frac{p'^2}{m^2}\right)\right]\right\}.\end{aligned} \tag{8.10}$$

Now we can compare our final results with the findings in [12]. For the scalar propagator, Eq. (8.5) is in complete agreement with the results quoted in [12], after taking into account the conventions of momentum flow. The same is true for the scalar-photon three-point vertex. Notice that in [12], this result is expressed in terms of nine inequivalent vector and tensor integrals which are $K^{(0)}$, $J^{(0)}$, $I^{(0)}$, $K^{(1)}_\mu$, $J^{(1)}_\mu$, $I^{(1)}_\mu$, $J^{(2)}_{\mu\nu}$, $I^{(2)}_{\mu\nu}$ and $I^{(3)}_{\mu\nu\alpha}$. In our analysis, the use of total derivative terms has allowed us to reduce the number of independent integrals by two; i.e., we do not require $I^{(2)}_{\mu\nu}$ and $I^{(3)}_{\mu\nu\alpha}$ to express the vertex.

IX. CONCLUSIONS

To summarize, in this paper we have applied the string-inspired worldline formalism to a number of interrelated issues in scalar QED:

(i) We have rederived the momentum-space Bern-Kosower type master formula for the tree-level scalar propagator dressed by an arbitrary number of photons, obtained in [36] by a comparison with Feynman parameter integrals, starting directly from the worldline path integral representation of this amplitude. We have also generalized this master formula to the x-space propagator.

(ii) We have used the master formula for constructing, by sewing in Feynman gauge, the one-loop scalar propagator and the one-loop vertex in arbitrary dimension.

(iii) These momentum-space results were extended to an arbitrary covariant gauge in a relatively simple way, observing that the difference terms involve only total derivatives under the worldline integrals. We have checked that the result agrees with the earlier calculation presented in [12].

(iv) In x-space, the implementation of changes of the gauge parameter through total derivatives has allowed us to obtain, in a very simple way, an explicit nonperturbative formula for the effect of such a gauge parameter change on an arbitrary amplitude summed to all loop orders. This formula generalizes the LKFT and contains it as a special case. At leading order in the ϵ-expansion it can be written in terms of conformal cross ratios.

(v) We have illustrated with an example how this nonperturbative transformation works diagrammatically in perturbation theory.

All this can be carried through quite analogously for spinor QED. Although the LKFT for the propagator has the same form in scalar and spinor QED [18], at higher points difference terms do arise, as will be discussed elsewhere [53]. Another extension of obvious interest is to the non-Abelian case. Using the worldline formalism along the lines of [54], a master formula for the scalar propagator dressed by external gluons has been obtained in [38]. As a next step, this could be used to construct the fully off-shell quark-gluon vertex and its Ball-Chiu form factor decomposition in any covariant gauge.

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APPENDIX A: A LIST OF INTEGRALS

Here we collect some integrals arising in the calculation of the propagator and vertex that permit simple closed-form expressions in terms of the hypergeometric function ${}_2F_1$ in an arbitrary dimension D. All of them are easily obtained using standard Feynman-Schwinger parameters,

$$
\begin{aligned}
\int \frac{d^Dq}{(2\pi)^D}\frac{1}{m^2+(q+p)^2} &= \frac{1}{(4\pi)^{\frac{D}{2}}}(m^2)^{\frac{D}{2}-1}\Gamma\left(1-\frac{D}{2}\right),\\
\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2[m^2+(q+p)^2]} &= -\frac{1}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)(m^2)^{\frac{D}{2}-2}{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right),\\
\int \frac{d^Dq}{(2\pi)^D}\frac{q^\mu}{q^2[m^2+(q+p)^2]} &= \frac{1}{2}\frac{\Gamma(1-\frac{D}{2})(m^2)^{\frac{D}{2}-2}p^\mu}{(4\pi)^{\frac{D}{2}}}\left\{\left(1+\frac{m^2}{p^2}\right){}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right)-\frac{m^2}{p^2}\right\},\\
\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^4[m^2+(q+p)^2]} &= \frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}(m^2)^{\frac{D}{2}-3}{}_2F_1\left(3-\frac{D}{2},2;\frac{D}{2};-\frac{p^2}{m^2}\right),\\
\int \frac{d^Dq}{(2\pi)^D}\frac{q^\mu}{q^4[m^2+(q+p)^2]} &= \frac{1}{2}\frac{\Gamma(1-\frac{D}{2})(m^2)^{\frac{D}{2}-2}p^\mu}{(4\pi)^{\frac{D}{2}}p^2}\left\{{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right)-\frac{(m^2+p^2)}{m^2}{}_2F_1\left(3-\frac{D}{2},2;\frac{D}{2};-\frac{p^2}{m^2}\right)\right\}.
\end{aligned}
\tag{A1}
$$

APPENDIX B: CALCULATION OF THE VERTEX

In this appendix we fill in some of the details of the calculation of the first and second vertex diagrams, Figs. 5(a) and 5(b). From Eqs. (8.6) and (8.7), the gauge part of $\Gamma_a[p,p';k_2,\varepsilon_2]$ can be written as

$$\begin{aligned}\Gamma^\mu_{a,\xi}[p,p';k_2,\varepsilon_2] \equiv & -e^3(m^2+p'^2)(m^2+p^2)\\ & \times\int_0^\infty dT\,\mathrm{e}^{-T(m^2+p^2)}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\\ & \times\int_0^{\tau_2} d\tau_3\int\frac{d^Dq}{(2\pi)^D}l_2^\mu\\ & \times\left[-(1-\xi)\frac{(l_1\cdot q)(l_3\cdot q)}{q^4}\right]\mathrm{e}^{c_1\tau_1-c_2\tau_2+c_3\tau_3}\\ = & \, e^3(1-\xi)(m^2+p'^2)(m^2+p^2)\\ & \times\int\frac{d^Dq}{(2\pi)^D}\frac{l_2^\mu}{q^4}\int_0^\infty dTT^3\mathrm{e}^{-T(m^2+p^2)}\\ & \times\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\frac{1}{T^2}\\ & \times\frac{\partial^2}{\partial u_1\partial u_3}\mathrm{e}^{T(c_1u_1-c_2u_2+c_3u_3)},\end{aligned} \tag{B1}$$

where $l_2 = k_2 + 2(p-q)$ and

$$\begin{aligned} c_1 &= 2p\cdot q - q^2,\\ c_2 &= 2k_2\cdot p + k_2^2 - 2q\cdot k_2,\\ c_3 &= q^2 + 2q\cdot p'. \end{aligned} \tag{B2}$$

The calculation of the parameter integrals is straightforward. Using

$$\begin{aligned} m^2+p^2-c_1 &= m^2+(p-q)^2,\\ m^2+p^2+c_2-c_1 &= m^2+(p'+q)^2,\\ m^2+p^2+c_2-c_1-c_3 &= m^2+p'^2, \end{aligned} \tag{B3}$$

the result can be written as

$$\begin{aligned}&\int_0^\infty dTT\,\mathrm{e}^{-T(m^2+p^2)}\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\frac{\partial^2}{\partial u_1\partial u_3}\mathrm{e}^{T(c_1u_1-c_2u_2+c_3u_3)}\\ &=\frac{1}{(m^2+p'^2)[m^2+(p-q)^2]}-\frac{1}{[m^2+(p-q)^2][m^2+(p'+q)^2]}-\frac{1}{(m^2+p'^2)(m^2+p^2)}+\frac{1}{(m^2+p^2)[m^2+(p'+q)^2]}.\end{aligned} \tag{B4}$$

Using this result together with (8.9) and (A1) one gets the final result for diagram 5(a) as given in Eq. (8.8).

Now, let us look at the second diagram of Fig. 5. From (8.6),

$$\begin{aligned}\Gamma^\mu_b[p,p';k_2,\varepsilon_2] = & -e^3(m^2+p^2)(m^2+p'^2)\int_0^\infty dTT^2\mathrm{e}^{-T(m^2+p^2)}\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\delta(u_1-u_2)\\ & \times\int\frac{d^Dq}{(2\pi)^D}\left\{-\frac{l_3^\mu}{q^2}+(1-\xi)\frac{q^\mu(l_3\cdot q)}{q^4}\right\}\mathrm{e}^{T(c_1u_1-c_2u_2+c_3u_3)},\end{aligned} \tag{B5}$$

where $l_3 = q+2p'$, and in the gauge part one can rewrite

$$(1-\xi)\frac{q^\mu(l_3\cdot q)}{q^4}\mathrm{e}^{T(c_1u_1-c_2u_2+c_3u_3)} = (1-\xi)\frac{q^\mu}{Tq^4}\frac{\partial}{\partial u_3}\mathrm{e}^{T(c_1u_1-c_2u_2+c_3u_3)}. \tag{B6}$$

The delta function kills one of the parameter integrals as

$$\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\delta(u_1-u_2)\mathrm{e}^{T(c_1u_1-c_2u_2+c_3u_3)} = \frac{1}{2}\int_0^1 du_1\int_0^{u_1}du_3\mathrm{e}^{T(c_1-c_2)u_1+Tc_3u_3}. \tag{B7}$$

The first term in the curly bracket after evaluating all the parameter integrals leads to

$$\begin{aligned} I_b^\mu &\equiv \int \frac{d^D q}{(2\pi)^D} \int_0^\infty dT T^2 \mathrm{e}^{-T(m^2+p^2)} \left[-\frac{q^\mu + 2p'^\mu}{q^2} \right] \int_0^1 du_1 \int_0^{u_1} du_3 \mathrm{e}^{T(c_1-c_2)u_1 + Tc_3u_3} \\ &= -\frac{1}{(m^2+p^2)(m^2+p'^2)} \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu + 2p'^\mu}{q^2[m^2+(p'+q)^2]}. \end{aligned} \tag{B8}$$

This implies that $I_b^\mu \sim p'^\mu$, so that I_b^μ can be reconstructed from $I_b \cdot p'$. Multiplying both sides by p'^μ we obtain

$$\begin{aligned} p' \cdot I_b &= -\frac{1}{(m^2+p^2)(m^2+p'^2)} \int \frac{d^D q}{(2\pi)^D} \frac{q\cdot p' + 2p'^2}{q^2[m^2+(p'+q)^2]} \\ &= -\frac{1}{(m^2+p^2)(m^2+p'^2)} \int \frac{d^D q}{(2\pi)^D} \left\{ \frac{1}{2}\left[\frac{1}{q^2} - \frac{1}{m^2+(p'+q)^2} - \frac{m^2-3p'^2}{q^2[m^2+(p'+q)^2]} \right] \right\}. \end{aligned} \tag{B9}$$

The first integral vanishes, while the second and third integrals have been given in (A1).

Coming to the gauge part of diagram 5(b), here we have to calculate

$$\begin{aligned} &(1-\xi) \int \frac{d^D q}{(2\pi)^D} \int_0^\infty dT T\, \mathrm{e}^{-T(m^2+p^2)} \frac{q^\mu}{q^4} \int_0^1 du_1 \int_0^{u_1} du_3 \frac{\partial}{\partial u_3} \mathrm{e}^{T(c_1-c_2)u_1 + Tc_3u_3} \\ &\quad = (1-\xi) \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu}{q^4} \left[\frac{1}{(m^2+p^2)(m^2+p'^2)} - \frac{1}{(m^2+p^2)[m^2+(p'+q)^2]} \right]. \end{aligned} \tag{B10}$$

Here again the first integral vanishes and the second one was given in (A1). Putting all this together one gets the final result for this diagram which was presented in Eq. (8.10).

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