Relativistic approach to the tunneling-time problem

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We report a study of the effect of relativity on the tunneling time for the process of the tunneling ionization. Our results indicate that the relativistic effects make the tunneling time nonzero. A time interval of a few attoseconds is required for the electron to traverse the barrier. This finding agrees with the basic postulate of the special relativity that two events which are in causal relationship cannot occur simultaneously.

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I. INTRODUCTION

Tunneling time in strong-field ionization of atoms is a fascinating area of research which received great impetus from the recent advances in experimental techniques. With the advent of the attoclock technique [1], the question of how exactly atomic tunneling ionization develops in time became an issue which can be directly addressed experimentally [1–3]. The attoclock technique relies on the use of a driving laser pulse with nearly circular polarization state to ionize the atom. A typical observable in the experiments using this technique is the offset angle θ defined as the location of the maximum of the electron momentum distribution in the polarization plane with respect to the major axis of the polarization ellipse.

Qualitative understanding of the principles behind the attoclock technique can be obtained on the basis of the Keldysh theory [4], its developments, and modifications [5–9]. For the tunneling regime characterized by the small values of the Keldysh parameter $\gamma = \omega\sqrt{2I}/E_0 < 1$ defined via the frequency ω, strength E_0 of the laser field, and the ionization potential I of the target atom, a well-known prediction [7,8] is that the maximum of the electron momentum distribution in the polarization plane is located at the point $\boldsymbol{p}_m = -\boldsymbol{A}(t_0)$. Here $\boldsymbol{A}(t)$ is the vector potential of the pulse, and t_0 is the moment of time when electric field of the pulse reaches it maximum value. This is a very useful result which lies at the basis of many semiclassical approaches describing the tunneling ionization process.

In the context of the attoclock technique, the semiclassical picture discussed above would imply that the offset angle should be zero. This prediction, however, is not entirely correct. The offset angle measured in the experiment can have nonzero value for two reasons. First, the interaction of the escaping electron and the ionic core potential may result in the deflection of the most probable electron trajectory and, consequently, in the nonzero value of the offset angle θ. Second, if we assume that an electron is ionized not at the moment of time t_0 of the maximum field strength, but rather at some moment of time $t_0 + \delta t$, the most probable electron trajectory will receive additional deflection because the value of the vector potential at the initial moment of time is different from $\boldsymbol{A}(t_0)$. The goal of the attoclock experiment is to infer the value of the tunneling delay δt from the experimentally observed value of the offset angle. This necessitates obtaining an accurate estimate of the role of the ionic core potential we mentioned above, which is indispensable for the correct interpretation of the experiment. Such an estimate can be obtained using the analytical R-matrix theory (ARM) [10,11], which allows one to extract the tunneling delay from the experimental value of the offset angle. This recipe was tested in our recent paper [12], where it was shown that the ARM results agree very well with the results given by the *ab initio* solution of the time-dependent Schrödinger equation. It was found further that the ionization time delays δt extracted from the values of the total offset angles and the offset angles due to the electron-core interaction had negative values for the range of intensities (up to several units of $10^{14}\,\mathrm{W/cm^2}$) [12]. The negative values for the ionization times can only be interpreted as zero tunneling delays for the process of the strong-field ionization of the hydrogen atom in the tunneling regime. Thus, according to these results, tunneling should be regarded as an instantaneous process.

This is a perfectly possible scenario in the nonrelativistic physics. Account of the special relativity effects, however, may modify this conclusion which seems to be at variance with its postulates. In particular, if we consider tunneling as a real physical process, which is marked by the events of electron entering the sub-barrier region and leaving this region (a picture which is often used to describe tunneling), then zero tunneling time presents a problem, since it is at variance with the basic postulate of the relativity theory that no two events which are in causal relationship can occur simultaneously. It has been pointed out [13–15] that correct treatment of the relativistic effects may prove very important for the determination of the tunneling time.

A relativistic semiclassical theory of atomic ionization has been given in [16]. Authors considered effects due to the relativistic kinematics and effects due to the interaction of electron spin with the magnetic field of the pulse using a relativistic generalization of the so-called imaginary time method (ITM) [17–19]. This approach allowed one to study relativistic effects for the ionization process driven by a pulse of arbitrary polarization and constant envelope [16]. The purpose of the present work is to provide relativistic description of a typical attoclock experiment, employing an ultrashort driving pulse, and to answer a question of what (if any) is the relativity effect on the tunneling time. We present

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results of a numerical simulation based on the solution of the time-dependent Dirac equation.

Before describing the method we employ, we should add that the very notion of the tunneling time is not completely free of a certain ambiguity (a comprehensive review of the different definitions of the tunneling time available in the literature is given in [20]). The view of ionization as a process occurring at a particular moment of time is based on the following picture provided by the ITM. Ionization process starts at some complex moment of time $t_1 + i\,t_2$ with electron having imaginary velocity. The trajectory giving the most probable electron momentum is a classical trajectory defined by these initial conditions. The complex character of the initial conditions and the trajectory itself reflects the classical impossibility of the tunneling process. In the complex t plane this trajectory is often assumed to consist of two segments. First, it descends perpendicularly to the real axis, so that $\mathrm{Re}t = t_1$. This part of the trajectory can be interpreted as the under-the-barrier motion. As trajectory intersects the real axis, the electron velocity becomes real and the second segment of the electron trajectory in this picture is an interval of the real t axis (t_1,t_f) (here t_f is the moment of time when electron arrives at the detector). This second segment corresponds to the after-the-barrier motion. The point t_1 on the real axis corresponds thus to an exit point; the values of the electron coordinates at this moment of time give coordinates of the point in space where electron exits from under the barrier. This picture is very appealing and is often used (explicitly or implicitly) in discussing tunneling ionization. One should note however that, strictly speaking, the very notion of the exit point is somewhat ill defined as emphasized in [19]. Indeed, it hinges on the choice of the particular integration path in the complex t plane which we described above. This path is used to calculate the classical action which gives the ionization probability. The action being an analytic function of complex t, this path can however be deformed (as long as it does not cross any singular points), which makes the notions of time and space coordinates of the exit point somewhat ambiguous. In view of this it might be argued that it would be more consistent to use the experimentally measured deflection angle θ to define the tunneling time.

Atomic units with $\hbar = 1$, $e = 1$, $m = 1$, and $c \approx 137.036$ (here e and m are charge and mass of the electron and c speed of light) are used throughout the paper.

II. THEORY

We solve the time-dependent Dirac equation:

$$i\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t} = (\hat{H}_{\rm atom} + \hat{H}_{\rm int})\Psi(\boldsymbol{r},t), \tag{1}$$

where

$$\hat{H}_{\rm atom} = c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + c^2(\beta - I) + I\,V(r) \tag{2}$$

is a field-free Hamiltonian and

$$\hat{H}_{\rm int} = c\boldsymbol{\alpha}\cdot\boldsymbol{A} \tag{3}$$

describes the atom in the field of the laser pulse. In the equations above $\Psi(\boldsymbol{r},t)$ is a four-component bispinor, $\boldsymbol{\alpha}$, and β are Dirac matrices; c is the speed of light. To make the comparison with the nonrelativistic calculations more transparent, we subtracted the constant term Ic^2 corresponding to the rest mass energy of the electron from the field-free atomic Hamiltonian in Eq. (2). We use the Coulomb gauge; the interaction of the atom and the external electromagnetic field is described by means of the vector potential $\boldsymbol{A}$ in Eq. (3). We consider below the ionization process driven by an ultrashort circularly polarized pulse propagating along the x direction which can be described by the vector potential:

$$\begin{aligned}\boldsymbol{A}_y(\boldsymbol{r},t) &= \frac{E_0\sqrt{2}}{\omega}T(\zeta)\sin^2\left(\frac{\pi\zeta}{T_1}\right)\cos(\omega\zeta),\\ \boldsymbol{A}_z(\boldsymbol{r},t) &= \frac{E_0\sqrt{2}}{\omega}T(\zeta)\sin^2\left(\frac{\pi\zeta}{T_1}\right)\sin(\omega\zeta),\end{aligned} \tag{4}$$

where $\zeta = t - x/c$, $T(\zeta)$ is a rectangular window function, such that $T(\zeta) = 1$ for $\zeta \in (0,T_1)$ and zero outside this interval, with T_1 being a total pulse duration, $\omega = 0.057$ a.u. is the base frequency, and E_0 is the field strength of the laser pulse.

To solve the time-dependent Dirac equation we employed the procedure described in detail in a recent paper [21].

Solution is sought as an expansion in basis bispinors:

$$\Psi(\boldsymbol{r},t) = \sum_{\substack{j\\ l=j\pm1/2}}\sum_{M=-j}^{j}\Psi_{jlM}(\boldsymbol{r},t), \tag{5}$$

where each basis bispinor is

$$\Psi_{jlM}(\boldsymbol{r},t) = \begin{pmatrix} g_{jlM}(r,t)\Omega_{jlM}(\boldsymbol{n})\\ f_{jlM}(r,t)\Omega_{jl'M}(\boldsymbol{n})\end{pmatrix}, \tag{6}$$

and two-component spherical spinors are defined as $\Omega_{jlM}(\boldsymbol{n}) = \begin{pmatrix} C^{jM}_{l\,M-\frac12\,\frac12\,\frac12}Y_{l,M-\frac12}(\boldsymbol{n})\\ C^{jM}_{l\,M+\frac12\,\frac12\,-\frac12}Y_{l,M+\frac12}(\boldsymbol{n})\end{pmatrix}$, where $C^{jM}_{lm\frac12\mu}$ are the Clebsch-Gordan coefficients, $Y_{lm}(\boldsymbol{n})$ the spherical harmonics, and $\boldsymbol{n} = \boldsymbol{r}/r$. Parameters l and l' in Eq. (5) must satisfy the relation $l + l' = 2j$.

The radial functions $g_{jlM}(r,t)$ and $f_{jlM}(r,t)$ in Eq. (5) and Hamiltonian operator $\hat{H}$ are discretized on the grid with the step size δr in a box of the size $R_{\rm max}$ (to be specified below). Substitution of the expansion (5) in the discretized Dirac equation gives a set of coupled equations on the radial amplitudes $g^n_{jlM}(t) = g_{jlM}(r_n,t)$, and $f^n_{jlM}(t) = f_{jlM}(r_n,t)$. To solve these equations we use a generalization of the well-known matrix iteration method (MIM) [22]. This well-known technique is often applied for the solution of the nonrelativistic TDSE for atoms in the laser field as was done in [23] or in our previous works [24–26]. Technical aspects of this generalization are described in detail in our recent paper [21]. Dependence of the vector potential in Eq. (4) on the spatial variables is treated by means of an expansion in spherical harmonics at every step during time integration as described in [21].

Differential ionization probabilities $P(\boldsymbol{p},\mu)$ (here $\boldsymbol{p}$ is the asymptotic electron momentum and μ the electron polarization state) are obtained by projecting the solution of the time-dependent Dirac equation after the end of the pulse on the set of the ingoing relativistic scattering states $\Psi^-_{\mu,\boldsymbol{p}}(\boldsymbol{r})$ of the

hydrogen atom [27]:

$$\Psi^{-}_{\mu,\boldsymbol{p}}(\boldsymbol{r}) = \sum_{jlM} i^{l} e^{-i\delta_{jl}(p)} \langle \Omega_{jlM}(\hat{\boldsymbol{p}}) | v_{\mu} \rangle \Psi_{pjlM}(\boldsymbol{r}), \qquad (7)$$

where $\Psi_{pjlM}(\boldsymbol{r})$ are continuous spectrum bispinor wave functions of the Dirac Hamiltonian, $\delta_{jl}(p)$ is the relativistic Coulomb phase shift, $\Omega_{jlM}(\hat{\boldsymbol{p}})$ is a two-component spherical spinor, and v_{μ} is a two-component spinor describing the electron polarization state.

As in the work in [12], we consider a neutral hydrogen atom as a target. For this target the relativistic effects are certainly quite small. To enhance the role of relativity we might consider a highly charged ion as a target. That, however, would increase the ionization potential dramatically and change the ionization regime completely. Our choice of the target system is motivated by the fact that our goal in the present paper is to answer the principal question if the relativity in any way affects the conclusion made in [12] of the instantaneous character of the tunneling process. We would like, therefore, to keep the laser pulse parameters close to the values used in [12] and routinely used in the attoclock experiments. We employed the following strategy. We consider the speed of light c in our calculations as a variable parameter, and perform the calculations for various values of c (still keeping it relatively large so that ionization potential changes insignificantly).

Using full Dirac equation for the neutral hydrogen where relativistic effects are certainly small might seem an overkill. We are certainly within the limits of the applicability of the perturbation theory considering relativistic effects as a perturbation. We could, therefore, try to tackle the problem by solving quantum evolution equation using the Breit-Pauli Hamiltonian [28], which includes the relativistic effects up to the order of v^2/c^2 (v is a typical electron velocity). Breit-Pauli Hamiltonian certainly contains all the physics necessary to consider in our problem. There is, however, one important technical advantage in using the Dirac Hamiltonian instead of the Breit-Pauli one. Some operators in the Breit-Pauli Hamiltonian are very singular. Consider, for example, the terms representing spin-orbit interaction and the so-called Darwin term [28]. For the hydrogen atom the spin-orbit interaction behaves as r^{-3} for small values of the radial coordinate. Such an operator is, strictly speaking, not self-adjoint, unless we narrow its domain of definition in the Hilbert space so that it acts only on the functions vanishing at the origin (otherwise its matrix elements may simply diverge due to the singular behavior of the integrand at the point $r = 0$). In the perturbative calculations using the Breit-Pauli Hamiltonian this problem does not arise, since selection rules due to angular integrations automatically ensure that with the proper choice of the basis of the zero-order wave functions (e.g., the wave functions of the nonrelativistic hydrogen atom) all radial integrals do converge. In the nonperturbative calculation, such as the solution of the time-dependent Breit-Pauli equation, it is more difficult to ensure self-adjointness of the spin-orbit term, which will have as an immediate effect the nonunitarity of the propagation operator. Similarly, the Darwin term (which is essentially a δ function of coordinates) is difficult to treat in a numerical calculation. For these reasons we found that solving the Dirac equation is technically simpler and gives more accurate and reliable results than solving the time-dependent Breit-Pauli evolution equation. We have performed the routine series of the accuracy checks in our calculation. The details of the numerical procedure used to obtain the results presented below were as follows. The Hamiltonian operator was discretized on a grid with the step size $\delta r = 0.1$ a.u.; the radial variable was restricted to an interval $(0, R_{\max})$, with $R_{\max} = 400$ a.u. Maximum value of the parameter j in Eq. (5) was $J_{\max} = 60\frac{1}{2}$. The initial hydrogen state was prepared by solving the eigenvalue problem for the discretized Hamiltonian, which resulted in the ground-state energy of $-0.500\,006\,661$ a.u., which is to be compared with the value $-0.500\,006\,657$ a.u. given by the Dirac formula (as we mentioned above, to facilitate comparison with the nonrelativistic calculations, we subtract the rest mass energy term mc^2 from the Hamiltonian). In the calculations of the tunneling ionization reported below we use the time step $\Delta t = 10^{-2}$ a.u. We have checked that variations of the essential parameters (the time step and parameters $R_{\max}$, $J_{\max}$ produce variations in the ionization probabilities of the order of 10^{-6} a.u., which is sufficient for the purposes of the present work).

To complete this section we will clarify how the integration method we used deals with the problems of the so-called Zitterbewegung [29] and the well-known problem of the collapse to the negative energies continuum. These problems are due to the absence of the lower bound for the field-free Dirac Hamiltonian operator (2). The negative energy states have energies vastly different form those with positive energies (here we refer to the usual definition of the Dirac Hamiltonian without subtraction of the rest mass energy term). If we have in the initial state a superposition of the states with positive and negative energies, we will have two vastly different time scales in the problem. To reproduce this phenomenon accurately we must use an integration time step $\Delta \ll 1/c^2$, which would make any practical calculation impossible. The problem here is similar to the problem of solving numerically stiff (the ones having very different time scales) systems of differential equations. Tiny features due to the small time scale can only be reproduced if an impractically small time step is used. Use of the stable integration method and reasonably small integration time step ensures, however, that while the tiny features cannot be reproduced, the overall behavior can be reproduced accurately. The MIM method we employ belongs to this class. No collapse to the negative energies continuum occurs in the course of the time propagation. To see this consider a simple model of a stiff system of two differential equations:

$$i\dot{\boldsymbol{x}} = \boldsymbol{A} \cdot \boldsymbol{x}, \qquad (8)$$

where $\boldsymbol{A}$ is a Hermitian matrix $\boldsymbol{A} = \mathrm{diag}(\lambda_1, \lambda_2)$, where λ_1 is of order of 1, while λ_2 has large negative value. Short-time propagator in the MIM method is a unitary Crank-Nicholson (CN) propagator [30]. Its application to the propagation of the solution of the system (8) gives

$$\boldsymbol{x}_{n+1} = \frac{1 - \frac{i\Delta}{2}\boldsymbol{A}}{1 + \frac{i\Delta}{2}\boldsymbol{A}} \boldsymbol{x}_n, \qquad (9)$$

where $\boldsymbol{x}_n$ is a solution at the nth step; Δ is an integration step. One can see that if at the nth step of the propagation

the second component of x_n acquires an error $\delta x_n^{(2)}$, the corresponding error $\delta x_m^{(2)}$ for $m > n$ remains bounded in the course of evolution, and there is no collapse to the negative energies continuum.

III. RESULTS

We are interested below in the electron momenta distribution in the polarization (y,z) plane of the laser pulse summed over electron polarization states. These distributions are shown in Fig. 1 for different values of the parameter c for the field intensity of 10^{14} W/cm^2.

Values of the offset angles, found as the maxima of the probability distributions in the polarization plane, are shown in Fig. 2 for two different sets of the laser pulse parameters in

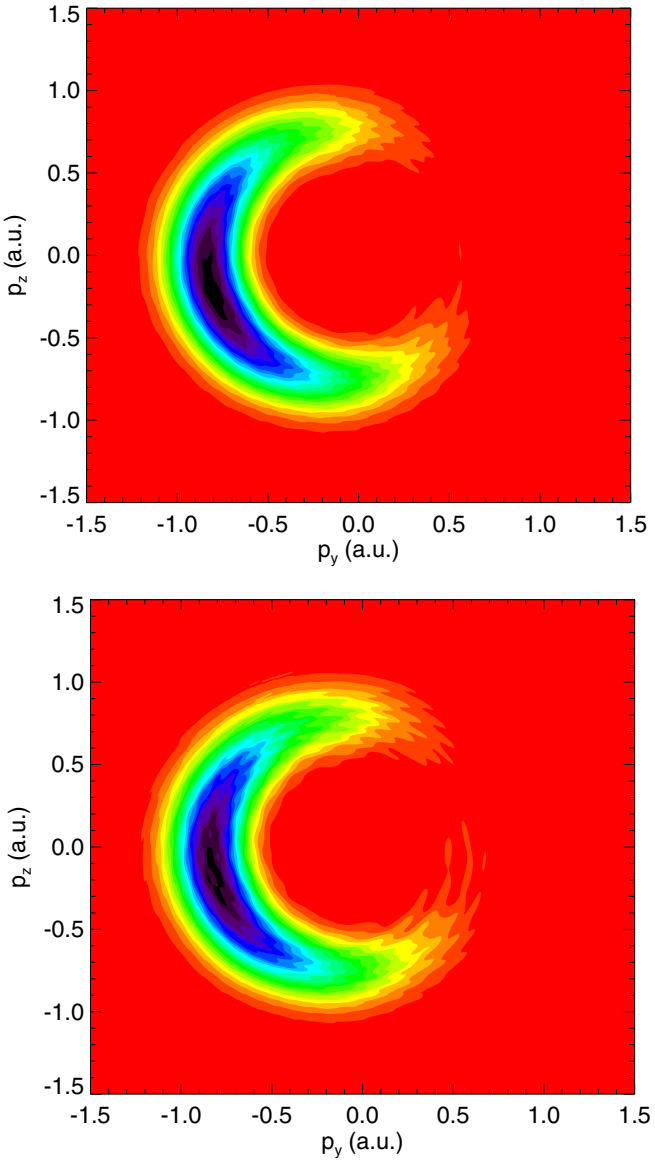

FIG. 1. (Color online) Electron momentum distribution in the polarization plane for the laser pulse (4) with pulse intensity of 10^{14} W/cm^2, pulse duration $T_1 = 2$ optical cycles, and speed of light $c = 137$ a.u. (top) and $c = 5$ a.u. (bottom).

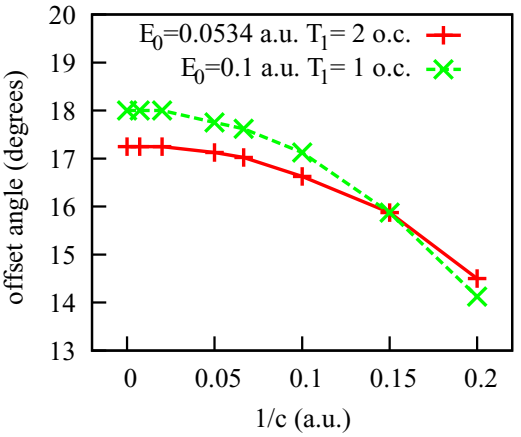

FIG. 2. (Color online) Offset angle as a function of the parameter $1/c$ for different pulse parameters in Eq. (4).

Eq. (4) as functions of the parameter c. The rationale behind our strategy of studying offset angles by varying c can be explained as follows.

The observable offset angle can be represented as a sum [12]:

$$\theta = \theta^t + \theta^{\rm int}, \tag{10}$$

where $\theta^t = w\tau$ is related directly to the tunneling delay τ, and $\theta^{\rm int}$ describes additional deflection of the electron trajectory which in the nonrelativistic theory is due to the electron-ion core interaction. In the relativistic theory both these terms depend on c, and for large enough c can be expanded into series in powers of a small quantity $1/c$. Let us consider such an expansion for $\theta^{\rm int}$ first:

$$\theta^{\rm int}(c) = \theta_0^{\rm int} + \frac{\theta_1^{\rm int}}{c} + \frac{\theta_2^{\rm int}}{c^2} + \cdots, \tag{11}$$

where the leading term $\theta_0^{\rm int}$ can be calculated using the methods of the ARM theory [10,11], as was done in [12]. To understand the origin of the higher-order corrections in Eq. (11) let us recapitulate briefly the procedure used to find the most probable electron momentum [10,31], which is directly related to the offset angle. To find the most probable electron momentum one has to find the most probable electron trajectory, which amounts to calculation of the classical action [31]. In the simplest variant of the theory [4] one has only to calculate the action for an electron in the presence of the electromagnetic field. The interaction between the escaping electron and the ion core can be taken into account by means of the so-called semiclassical perturbation theory [31], which consists in adding Coulomb correction to the action, considering it as a perturbation. Alternatively, one can use the so-called method of the imaginary time [18], or the ARM theory [10,11], which treat Coulomb interaction nonperturbatively. Implementing these procedures, one can obtain the contribution due to the electron-ion interaction effects.

In the relativistic theory one can employ a similar procedure using, of course, relativistic expression for the action [16]. In this approach the relativistic corrections originate from two different effects. One is the effect of the relativistic kinematics.

An important fact is that this correction is of the order of c^{-2} [32]. Corrections of the order of c^{-1} in Eq. (11) are due to the contribution of the nondipole terms arising from expansion of the argument of the vector potential in the Hamiltonian (3). This correction, however, vanishes in the polarization plane. One way to see it is by noting that this correction is linear in the photon momentum $\boldsymbol{k}$. The only scalar quantity linear $\boldsymbol{k}$ we can form from the available vectors is a dot product of $\boldsymbol{k}$ and electron momentum $\boldsymbol{p}$, which vanishes for $\boldsymbol{p}$ in the polarization plane.

We are led to the conclusion, therefore, that the coefficient $\theta_1^{\rm int}$ in Eq. (11) is zero, and thus the leading-order relativistic correction to $\theta^{\rm int}(c)$ is of the order of $1/c^2$.

Let us consider now the tunneling-time contribution to the observable offset in Eq. (10). Expanding $\theta^t(c)$ as a series in $1/c$ we can write

$$\theta^t(c) = \theta_0^t + \frac{\theta_1^t}{c} + \cdots . \tag{12}$$

The coefficient θ_0^t in this expansion is the nonrelativistic value of the tunneling-time contribution to the offset angle. It was found in [12] that for the field intensities of the order of $10^{14}\,\mathrm{W/cm^2}$ $\theta_0^t \approx 0$. An estimate for the coefficient θ_1^t in Eq. (12) can be easily obtained in this case if we use the picture employed in the original paper by Keldysh [4]. Let us describe briefly this scenario following the work in [33]. A tunneling electron has to traverse a barrier with the width $a_k \approx I/E_0$ (where I is the ionization potential and E_0 is the field strength), which is obtained from simple classical estimation. Assuming that electron velocity during the tunneling is of the order of $v_k \approx \sqrt{I/2}$, one obtains the so-called Keldysh time $\tau_k = \sqrt{2I}/E_0$. In this picture, the Keldysh parameter γ can be introduced in a natural way as a ratio of the Keldysh time and the period of the electromagnetic field, which explains the fundamental connection between the value of the Keldysh parameter and the profound difference of the ionization processes in multiphoton ($\gamma \gg 1$) and tunneling ($\gamma \ll 1$) regimes. More importantly, for the purpose of the present paper, the Keldysh time can be regarded as a lower limit for the tunneling time [33]. Assuming that electron velocity during tunneling cannot exceed the speed of light, we obtain an estimate for the tunneling delay δt:

$$\delta t \approx \tau_k = \frac{a_k}{v_k} > \frac{a_k}{c} = \frac{v_k}{c}\tau_k. \tag{13}$$

The corresponding lower limit for the tunneling time contribution to the value of the offset angle will then be

$$\theta^t(c) \approx \omega\delta t > \frac{v_k}{c}\gamma \approx \frac{\sqrt{I}}{c}\gamma, \tag{14}$$

where γ is the Keldysh parameter. From this expression one obtains an estimate $\theta_1^t \approx \sqrt{I}\gamma$ for the coefficient with the first-order term in the expansion (12). For the field parameters we use $\gamma \approx 1$, so for the hydrogen atom we obtain $\theta_1^t \approx 1$. From Eq. (10), Eq. (11), and Eq. (12) we see that the leading-order relativistic correction for the total observable offset angle $\theta(c)$ is of the order of $1/c$ and is solely due to the tunneling time effects. If, therefore, we find this leading-order relativistic correction, we determine the relativistic correction to the tunneling delay. A more refined semiclassical picture was given in [34], where it was shown that the Lorentz force due to the magnetic field of the laser pulse has an effect on the under-the-barrier motion and consequently on the tunneling time. Leading-order effect of the Lorentz force is clearly of the order of $1/c$, and it provides a classical explanation of the appearance of the term linear in $1/c$ in the expansion (12).

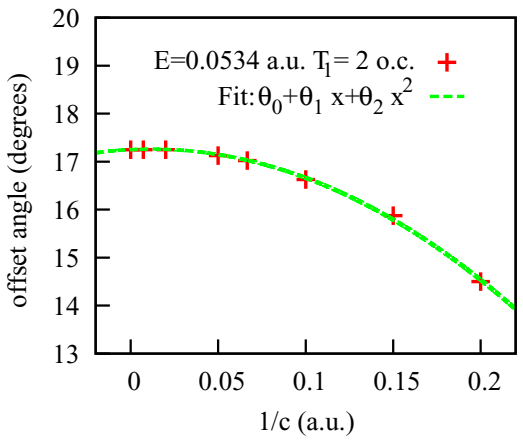

FIG. 3. (Color online) Offset angle as a function of the parameter $x = 1/c$ and results of a fitting procedure.

Information about the higher-order terms of the expansion (11) can be obtained by analyzing the behavior of the offset angle as a function of c. This behavior is shown in Fig. 2 for two different sets of the driving pulse parameters (we studied pulses with different parameters to make sure that our results are quite general).

A simple numerical analysis allows us to obtain an estimate of the value of the relativistic tunneling time from these data. We can find numerical estimates for the first derivatives of $\theta(x)$, where $x = 1/c$, at the point $x = 0$ using divided differences, or we can fit the function $\theta(x)$ near the point $x = 0$ using the polynomial fitting expression. Both procedures give similar results. As an illustration we present in Fig. 3 the results of a least-square fit of the function $\theta(x)$ based on a fitting formula $\theta_0 + \theta_1^t x + \theta_2 x^2$, where we treat θ_1^t and θ_2 as fitting parameters. Parameter θ_0 in this expansion was obtained from the nonrelativistic calculation for the same field geometry which we performed separately using the procedure for the solution of the nonrelativistic time-dependent Schrödinger equation described in [25]. Offset angle data from the interval $x \in (0,0.1]$ have been used in the fitting procedure. This procedure is stable in the sense that the resulting values of the coefficients vary only slightly if a different fitting interval is used. The least-square fit provides us with useful estimates of the fitting error. The fitting procedure gives the following results for the coefficient θ_1^t in the expansion (12). For the pulse parameters shown in Fig. 2 we obtain $\theta_1^t = 1.17 \pm 0.33$ a.u. (field strength $E_0 = 0.1$ a.u., and the pulse duration $T_1 = 1$ optical cycles) and $\theta_1^t = 1.81 \pm 0.52$ a.u. (field strength $E_0 = 0.0534$ a.u., and the pulse duration $T_1 = 2$ optical cycles). These values are compatible with the estimate for the value of θ_1^t we made above. Using these values for θ_1^t we obtain for the physical value $c = 137.036$ of the speed of light the following values for the relativistic corrections to the tunneling time: $\tau = 3.62 \pm 1.02$ as (field strength $E_0 = 0.1$ a.u., and the

pulse duration $T_1 = 1$ optical cycles), and $\tau = 5.61 \pm 1.61$ as (field strength $E_0 = 0.0534$ a.u., and the pulse duration $T_1 = 2$ optical cycles).

It is important to note that these corrections are positive. For the field strength of $E_0 = 0.0534$ a.u. corresponding to the intensity of $10^{14}\,\mathrm{W/cm^2}$ the leading nonrelativistic term in the expansion (12) for the tunneling time was found to have a value very close to zero [12]. Positive first-order relativistic correction makes the tunneling time in this case positive, with the total value which can be estimated as $\tau \approx 5$ as. For the intensities higher than $1.5 \times 10^{14}\,\mathrm{W/cm^2}$ the leading-order nonrelativistic term in the expansion (12) becomes negative and large enough [12] to make the total time delay for the field strength of $E_0 = 0.1$ (intensity of $1.5 \times 10^{14}\,\mathrm{W/cm^2}$) negative. For such field strengths, however, we may have another physical effect at play, the so-called frustrated tunneling [35,36]. This was the interpretation of the negative ionization times for high intensities suggested in [12]. If this interpretation is valid, this additional physical effect may hide the small relativistic effect we are investigating.

In summary, we performed a study of the effect of relativity on the tunneling time for the process of tunneling ionization. The deflection angles of the electron wave packets were obtained by solving the time-dependent Dirac equation in which the relativistic effects are properly taken into account. The scaling of the speed of light leads us to the conclusion that there is a nonzero time delay for the process of the tunneling ionization for the field intensity of $10^{14}\,\mathrm{W/cm^2}$. This finding provides insight into the process of the tunneling ionization. It agrees with the basic postulate of the special relativity that two events which are in causal relationship (and the events of electron entering and exiting the barrier certainly are) cannot occur simultaneously. Nonrelativistic treatment resulted in zero tunnneling-time delay [12]. That may be explained by the fact that the nonrelativistic calculation inevitably includes trajectories in which the velocity of the electron can be faster than the speed of light. More detailed study of the relativistic effects could be done using the relativistic modification of the ARM theory which, to our knowledge, is not available yet.

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